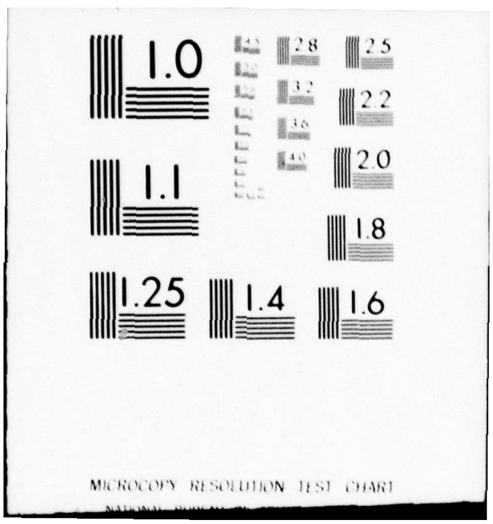
AD-A060 800 CALIFORNIA UNIV BERKELEY DEPT OF MECHANICAL ENGINEERING F/G 20/11
ON THERMAL EFFECTS IN THE THEORY OF SHELLS.(U)
SEP 78 A E GREEN, P M NAGHDI N00014-75-C-0148
UNCLASSIFIED UCB/AM-78-4 NL

| OF |

AD
A060 800

END
DATE
FILMED
1-79
DDC


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6
MICROCOPY RESOLUTION TEST CHART

9 11

OFFICE OF NAVAL RESEARCH

15 Contract NØØ14-75-C-Ø148

Project NR 064-436

14 Report No. UCB/AM-78-4

6 On Thermal Effects in the Theory of Shells,

by

9 Technical rept.,

10 A. E. Green [redacted] P. M. Naghdi

11 Sep[redacted] 1978

12 54p.

Department of Mechanical Engineering

University of California

Berkeley, California

On Thermal Effects in the Theory of Shells

by

A. E. Green[†] and P. M. Naghdi[‡]

Abstract. This paper is concerned with thermomechanics of thin shells by a direct approach based on the theory of a Cosserat surface comprising a two-dimensional surface and a single director attached to every point of the surface. In almost all previous developments of the thermo-mechanical theory of shells by direct approach, only one temperature field has been admitted. This allows for the characterization of temperature changes along some reference surface, such as the middle surface, of the (three-dimensional) shell-like body, but not for temperature changes along the shell thickness. A main purpose of the present study is to incorporate the latter effect into the theory; and, in the context of the theory of a Cosserat surface, this is achieved by a recent approach to thermomechanics (Green and Naghdi 1977) which provides a natural way of introducing two (or more) temperature fields at each material point of the surface. Apart from full discussion of thermomechanics of shells and thermodynamical restrictions arising from the second law of thermodynamics for shells, attention is given to a discussion of symmetries (including material symmetries) and thermal effects in the nonlinear theory of elastic shells with detailed discussion of the linear theory of elastic plates.

[†]Mathematical Institute, Oxford OX1 3LB, U.K.

[‡]Department of Mechanical Engineering, University of California, Berkeley, California 94720, U.S.A.

1. Introduction

This paper is concerned with thermomechanics of thin shells by a direct approach based on the theory of Cosserat (or directed) surfaces. A Cosserat surface is a body $\mathcal{C}$ comprising a two-dimensional surface (embedded in a Euclidean 3-space) and a single director (i.e., deformable vector) attached to every point of the surface.* A comprehensive account of the thermodynamical theory of a Cosserat surface -- hereafter designated $\mathcal{C}$ -- and its application to shell theory, together with an historical survey and a large number of relevant references is contained in an article by Naghdi (1972). For clarity's sake, we may recall that the material surface of $\mathcal{C}$ can be identified with a particular reference surface (often taken to be an interior surface) in the three-dimensional shell-like body, e.g., the middle surface of the shell in some fixed reference configuration; the director at each point is regarded as representing the material filament across the reference surface; and the component of the director along the normal of the reference surface can be taken as a measure of shell thickness.

Throughout our previous developments of the thermo-mechanical theory of shells by direct approach (Green *et al*. 1965, Green and Naghdi 1970, Naghdi 1972), only one temperature field has been admitted and this allows for the characterization of temperature changes along the reference surface of the shell-like body. Some indication of how temperature changes *across* the reference surface of the shell-like body could be dealt with has been given in the papers of Naghdi (1964) and of Green and Naghdi (1970,1971) by using three-dimensional approximations. One author, Zhilin (1976), has considered two temperature fields in a direct theory and we refer again to this paper below.

---

*The body $\mathcal{C}$ is taken to model some of the properties of a three-dimensional body of shell-like character. When the director is absent it reflects the properties of a material surface which can be the bounding surface between two different bodies or a surface in free space.

Recently mechanical and thermo-mechanical theories of a membrane, regarded as a two-dimensional surface in a Euclidean 3-space, have been published by Gurtin and Murdoch (1974,1975) and by Murdoch (1976a,b), where the surface is mainly regarded as a bounding surface of a body or as an interface between ambient media in which diffusion through the interface is neglected. Although these authors have placed special emphasis on residual stress and surface tension in the membrane surface, apart from notation, all their basic equations are special cases of those given previously. Murdoch (1976a), however, modified the entropy inequality used in earlier work in order to account for ambient temperatures in the surrounding media which may be different from each other and from the temperature of the interface, but with the temperature being everywhere continuous. In a second paper, Murdoch (1976b) seeks to show that his inequality is in line with three-dimensional considerations of Green and Naghdi (1970), but the present authors believe that the new inequality given by Murdoch is unsatisfactory, especially when only one temperature field is allowed for the interface. It appears that there may be confusion between boundary values of temperature and entropy with those field quantities which should enter any entropy inequality. We believe that the type of problem considered by Murdoch can only be discussed satisfactorily on the basis of a two-dimensional model for the interface if more than one temperature field is admitted. In the context of a direct approach based on a Cosserat surface, Zhilin (1976) formulates a theory in which two temperature fields are admitted; and he postulates two entropy inequalitites, but only one energy equation for the surface. Since this provides only one field equation for the two temperatures, Zhilin arbitrarily rewrites the energy field equation as a set of two differential equations. The physical basis for these two equations, which have not been obtained from any clearly stated balance laws, is obscure. It is difficult to see any relation between the work of Zhilin and that discussed in the present paper.

Although widespread use of the Clausius-Duhem inequalities has been made in three, two and one-dimensional continuum thermodynamics, these inequalities have been subject to the criticism that in some circumstances they do not reflect adequately ideas associated with the Second Law of Thermodynamics. Green and Naghdi (1977) have developed a new approach to three-dimensional continuum thermomechanics which is independent of any particular mathematical expression of the second law and which imposes some restrictions on the constitutive assumptions leading to a reduction of a number of independent response functions (or functionals) in the set of constitutive assumptions. In the present paper the same approach is used for the Cosserat surface and this provides a natural way of introducing two (or more) temperature fields.[+] When the director is absent, the theory reduces to that of a material surface which may be a surface in free space or a material surface between two different media. The contrast between the present theory and that of Murdoch (1976a) is illustrated by an example in §10. On the other hand, for an elastic material with one director and only one temperature field, we recover all the previous two-dimensional results. In addition, when we admit two temperature fields, results for an elastic plate agree with those found previously by Green and Naghdi (1970) from three-dimensional considerations.

Specifically, the contents of the paper are as follows. Section 2 contains a concise summary of the various basic results of the purely mechanical theory of a Cosserat surface with a single director. With reference to thermal properties, in §3 we admit at each material point of the surface of $\mathcal{C}$ a number of different two-dimensional temperatures and different two-dimensional entropies, as well as related thermal fields; and, in parallel with two-dimensional

[+]For the purely mechanical theory, it is already known how to extend the theory with more than one director; see, e.g., Green and Naghdi (1976).

conservation laws for balances of mass and momenta, we postulate balances of entropy. Next, we recall the balance of energy for the Cosserat surface; and, following the recent approach of Green and Naghdi (1977), after elimination of the assigned fields -- i.e., assigned force, assigned director force and external rates of supply of entropy -- regard the resulting equation as an identity to be satisfied for all thermo-mechanical processes. In §4, we briefly discuss thermoelastic theory of a Cosserat surface on the basis of the new procedure in thermomechanics (see §3) and also compare the results with earlier developments (see Green et al. 1965; Naghdi 1972) involving only a single temperature.

A new inequality representing the second law of thermodynamics for shells based on the present authors' earlier work (Green and Naghdi 1977,1978), along with restrictions on heat flux vectors and the specific internal energy are obtained in §5 and §8, respectively, while §§6-7 contain a discussion of relevant results for shells obtained from the three-dimensional theory. The last two sections (§§9-10) are devoted to a discussion of symmetries (including material symmetries) for shells and the linear thermoelastic theory of isotropic plates. The developments in §§9-10 supplement our earlier results by direct approach (Green and Naghdi 1970,1971) for thermoelastic shells in the presence of a single temperature.

The general theory given here is immediately available for problems in which the effect of surface tension and interfacial energies in a membrane surface are important, but we do not consider such problems in detail. Further discussion is necessary if there is diffusion across an interface, both in the context of the membrane theory and the theory of a Cosserat surface.

2. Summary of mechanical theory.

We summarize in this section the main kinematics and the basic equations of the purely mechanical theory of a Cosserat surface $\mathcal{C}$ and refer the reader to Naghdi (1972) for details and additional references on the subject. Let the particles of the material surface of $\mathcal{C}$ be identified with a system of convected coordinates θ^α $(\alpha = 1,2)$ and let the surface of $\mathcal{C}$ in the present configuration at time t, hereafter referred to as $\mathscr{s}$, occupy a two-dimensional region of space $\mathcal{R}$ bounded by a closed curve $\partial\mathcal{R}$. Similarly, in the present configuration, an arbitrary material surface of $\mathcal{C}$ occupies a portion of the two-dimensional region $\mathcal{R}$, which we denote by $\mathcal{P}$ $(\subseteq \mathcal{R})$ bounded by a closed curve $\partial\mathcal{P}$. Let $\underset{\sim}{r}$ and $\underset{\sim}{d}$ -- each a function of θ^α and t -- denote, respectively, the position vector of a typical point of $\mathscr{s}$ relative to a fixed origin and the director at $\underset{\sim}{r}$. Then, the base vectors along the θ^α-curves on $\mathscr{s}$ are defined by

$$\underset{\sim}{a}_\alpha = \underset{\sim}{a}_\alpha(\theta^\beta,t) = \partial\underset{\sim}{r}/\partial\theta^\alpha \quad , \tag{2.1}$$

and we denote the unit normal to $\mathscr{s}$ by $\underset{\sim}{a}_3(\theta^\alpha,t)$. Also

$$\begin{gathered} a_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta \quad , \quad \underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}_\beta = \delta^\alpha_\beta \quad , \quad a^{\alpha\beta} = \underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}^\beta \quad , \\ a = \det a_{\alpha\beta} \quad , \quad [\underset{\sim}{a}_1 \underset{\sim}{a}_2 \underset{\sim}{a}_3] > 0 \quad , \quad a^{\frac{1}{2}}\underset{\sim}{a}_3 = \underset{\sim}{a}_1 \times \underset{\sim}{a}_2 \quad , \end{gathered} \tag{2.2}$$

where δ^α_β is the Kronecker delta and $\underset{\sim}{a}^\alpha$ are contravariant base vectors.

A motion of the Cosserat surface is defined by

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^\alpha,t) \quad , \quad \underset{\sim}{d} = \underset{\sim}{d}(\theta^\alpha,t) \tag{2.3}$$

and we assume that $\mathbf{d}$ is nowhere tangent to $\mathcal{S}$, so that $\mathbf{a}_3 \cdot \mathbf{d} \neq 0$. The velocity and the director velocity vectors are given by

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{w} = \dot{\mathbf{d}} \quad , \tag{2.4}$$

where a superposed dot denotes differentiation with respect to t, holding θ^α fixed. Throughout this paper we use standard vector and tensor notations (see Naghdi 1972). Greek indices take the values 1,2 and the usual summation convention over a Greek superscript and a subscript is employed.

Consider now a reference configuration, which we take to be the initial configuration, of the Cosserat surface $\mathcal{C}$. Let the reference surface in this configuration be referred to by $\mathcal{S}$ with $\mathbf{R}$ as its position vector; let $\mathbf{A}_\alpha$ and $\mathbf{A}_3$ denote, respectively, the base vectors along the θ^α-curves on $\mathcal{S}$ and the unit normal to $\mathcal{S}$; and let $\mathbf{D}$ be the reference director at $\mathbf{R}$. Then,

$$\mathbf{R} = \mathbf{R}(\theta^\alpha) = \mathbf{r}(\theta^\alpha, 0) \quad , \quad \mathbf{D} = \mathbf{D}(\theta^\alpha) = \mathbf{d}(\theta^\alpha, 0) \quad , \quad \mathbf{A}_\alpha = \partial \mathbf{R}/\partial \theta^\alpha \tag{2.5}$$

and

$$\begin{gathered} A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta \quad , \quad \mathbf{A}^\alpha \cdot \mathbf{A}_\beta = \delta^\alpha_\beta \quad , \quad A^{\alpha\beta} = \mathbf{A}^\alpha \cdot \mathbf{A}^\beta \quad , \\ A = \det A_{\alpha\beta} \quad , \quad [\mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3] > 0 \quad , \quad A^{\frac{1}{2}} \mathbf{A}_3 = \mathbf{A}_1 \times \mathbf{A}_2 \quad . \end{gathered} \tag{2.6}$$

We assume that the kinetic energy per unit area of the surface of $\mathcal{C}$ in the present configuration is given by

$$T = \tfrac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v} + 2k^1 \mathbf{v} \cdot \mathbf{w} + k^2 \mathbf{w} \cdot \mathbf{w}) \quad , \tag{2.7}$$

where $\rho = \rho(\theta^\gamma, t)$ is the mass per unit area of $\mathcal{S}$ and the inertia coefficients k^1, k^2 are functions of θ^γ and independent of t. We define momenta corresponding to $\mathbf{v}$ and $\mathbf{w}$ as

$$\frac{\partial T}{\partial \underset{\sim}{v}} = \rho(\underset{\sim}{v} + k^1\underset{\sim}{w}) \quad , \quad \frac{\partial T}{\partial \underset{\sim}{w}} = \rho(k^1\underset{\sim}{v} + k^2\underset{\sim}{w}) \quad , \tag{2.8}$$

respectively, per unit area of $\mathcal{J}$.

With reference to the present configuration at time t, for each part $\mathcal{P}$ of $\mathcal{J}$, we postulate the equations of mass conservation, momentum, director momentum and moment of momentum as follows:

$$\frac{d}{dt}\int_{\mathcal{P}} \rho d\sigma = 0 \quad , \tag{2.9}$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\underset{\sim}{v} + k^1\underset{\sim}{w})d\sigma = \int_{\mathcal{P}} \rho\underset{\sim}{f}d\sigma + \int_{\partial\mathcal{P}} \underset{\sim}{N}ds \quad , \tag{2.10}$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(k^1\underset{\sim}{v} + k^2\underset{\sim}{w})d\sigma = \int_{\mathcal{P}} (\rho\underset{\sim}{\ell} - \underset{\sim}{m})d\sigma + \int_{\partial\mathcal{P}} \underset{\sim}{M}ds \quad , \tag{2.11}$$

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}} \rho\{\underset{\sim}{r} \times (\underset{\sim}{v} + k^1\underset{\sim}{w}) + \underset{\sim}{d} \times (k^1\underset{\sim}{v} + k^2\underset{\sim}{w})\}d\sigma \\ = \int_{\mathcal{P}} \rho(\underset{\sim}{r} \times \underset{\sim}{f} + \underset{\sim}{d} \times \underset{\sim}{\ell})d\sigma + \int_{\partial\mathcal{P}} (\underset{\sim}{r} \times \underset{\sim}{N} + \underset{\sim}{d} \times \underset{\sim}{M})ds \quad .\end{aligned} \tag{2.12}$$

In (2.10) to (2.12), $\underset{\sim}{N} = \underset{\sim}{N}(\theta^\gamma, t ; \underset{\sim}{\nu})$ is the force vector and $\underset{\sim}{M} = \underset{\sim}{M}(\theta^\gamma, t ; \underset{\sim}{\nu})$ the director force vector† at the curve $\partial\mathcal{P}$, where $\underset{\sim}{\nu}$ is the outward unit normal to $\partial\mathcal{P}$ and

$$\underset{\sim}{\nu} = \nu_\alpha \underset{\sim}{a}^\alpha \quad . \tag{2.13}$$

The vector field $\underset{\sim}{m}$ in (2.11) is the internal director force per unit area of $\mathcal{J}$ and it makes no contribution to the moment of momentum equation. Also $\underset{\sim}{f} = \underset{\sim}{f}(\theta^\gamma, t)$ is the assigned force vector and $\underset{\sim}{\ell} = \underset{\sim}{\ell}(\theta^\gamma, t)$ is the assigned director force vector per unit mass. These include contributions which model the action of forces over the major surfaces of a shell.

---

†The vector $\underset{\sim}{N}$, which represents force per unit length, has the dimension $[MT^{-2}]$ where [M] and [T] stand for the physical dimensions of mass and time. If the director $\underset{\sim}{d}$ has the dimension of length then k^1, k^2 are dimensionless and $\underset{\sim}{M}$ has the same dimension as $\underset{\sim}{N}$. If $\underset{\sim}{d}$ is dimensionless then $\underset{\sim}{M}$ has the dimension $[MLT^{-2}]$ where [L] is the physical dimension of length. In the latter case, $\underset{\sim}{M}$ is sometimes called a director couple.

Under suitable continuity assumptions the curve force vector $\mathbf{N}$ and the director force vector $\mathbf{M}$ can be expressed as

$$\mathbf{N} = \mathbf{N}^{\alpha}\nu_{\alpha} \quad , \quad \mathbf{M} = \mathbf{M}^{\alpha}\nu_{\alpha} \quad , \tag{2.14}$$

where $\mathbf{N}^{\alpha}, \mathbf{M}^{\alpha}$ transform as contravariant surface vectors. The local field equations corresponding to (2.9) to (2.12) are then

$$\rho a^{\frac{1}{2}} = \lambda(\theta^{\gamma}) \quad \text{or} \quad \dot{\rho} + \rho \mathbf{a}^{\alpha} \cdot \mathbf{v}_{,\alpha} = 0 \quad , \tag{2.15}$$

$$(a^{\frac{1}{2}}\mathbf{N}^{\alpha})_{,\alpha} + \lambda \mathbf{f} = \lambda(\dot{\mathbf{v}} + k^{1}\dot{\mathbf{w}}) \quad , \tag{2.16}$$

$$(a^{\frac{1}{2}}\mathbf{M}^{\alpha})_{,\alpha} + \lambda \boldsymbol{\ell} = \mathbf{m} a^{\frac{1}{2}} + \lambda(k^{1}\dot{\mathbf{v}} + k^{2}\dot{\mathbf{w}}) \quad , \tag{2.17}$$

$$\mathbf{a}_{\alpha} \times \mathbf{N}^{\alpha} + \mathbf{d} \times \mathbf{m} + \mathbf{d}_{,\alpha} \times \mathbf{M}^{\alpha} = \mathbf{0} \quad , \tag{2.18}$$

where a comma denotes partial differentiation.

In the absence of the director, the field equations for the surface are reduced to (2.15) together with

$$(a^{\frac{1}{2}}\mathbf{N}^{\alpha})_{,\alpha} + \lambda \mathbf{f} = \lambda \dot{\mathbf{v}} \quad , \quad \mathbf{a}_{\alpha} \times \mathbf{N}^{\alpha} = \mathbf{0} \quad . \tag{2.19}$$

These are one form of membrane equations of motion. The component forms of these equations were noted by Green _et al_. (1965, §7) and were derived from an energy equation together with invariance conditions under superposed rigid body motions. For a more general derivation using invariance conditions and a further discussion of the membrane theory, see Naghdi (1972, pp. 487-490 and pp. 546-547). If an interface between two ambient media is regarded to be a membrane surface, then the membrane equations for the interface must be supplemented of course by appropriate equations for each media.

3. Thermal properties. Thermodynamical theory of shells.

In most existing works on the theory of a Cosserat surface only one temperature field is admitted and this is regarded as representing the temperature variation in some reference surface, such as the middle surface, of the shell-like body. Also, the effect of the thermal boundary conditions on the major surfaces of the shell-like body[†] are incorporated into the theory through the external surface rate of supply of heat. The variations of the temperature along the shell thickness have not been modelled so far by a direct approach (within the scope of the theory of Cosserat surfaces), although some indications of how this could be effected is implicit in some work on thermoelastic shells from the three-dimensional equations by the present authors (Naghdi 1964, Green and Naghdi 1970) and in the monograph by Naghdi (1972). As already noted in §1, because of the new approach to thermomechanics of continua introduced recently (Green and Naghdi 1977,1978), it is now possible to account in a more general manner for the thermal properties of a shell-like body in the direct formulation of the theory based on a Cosserat surface.

Thus, at each material point of the material surface of $\mathcal{C}$, we introduce the scalar fields[‡] $\theta = \theta(\theta^\gamma, t)$ and $\theta_N = \theta_N(\theta^\gamma, t)$, $(N = 1, 2, \ldots, K)$, representing the effects of the temperature variation in a shell-like body: the surface temperature θ, which we require to be positive, represents the absolute temperature in the reference surface of the shell-like body, while the scalars θ_N account for the temperature variations along the thickness of the shell. In addition to the temperatures θ and θ_N, we admit the existence of[*] external rates of supply of heat $r = r(\theta^\gamma, t)$, $r_N = r_N(\theta^\gamma, t)$ per unit mass of $\mathcal{S}$ and external rates of curve supply of heat $-\bar{h}$, $-\bar{h}_N$ per unit length acting across the boundary $\partial\mathcal{R}$. Also, we

[†] The terminology of major surfaces refers to the upper and lower surfaces of the body separated by the thickness of the shell.

[‡] No confusion should arise from the use of the symbol θ in the designation of the temperature fields by $\theta, \theta_1, \theta_2, \ldots, \theta_K$ and the notation $\theta^\gamma = (\theta^1, \theta^2)$ for the convected coordinates.

[*] The external rates of supply of heat r and r_N include contributions corresponding to heat fluxes on the major surfaces of the shell. They are not the same as quantities defined with a similar notation in Green and Naghdi (1970) or in Naghdi (1972).

assume the existence of internal curve fluxes of heat $-h = -h(\theta^\gamma, t\,;\,\underset{\sim}{\nu})$, $-h_N = -h_N(\theta^\gamma, t\,;\,\underset{\sim}{\nu})$ across each curve $\partial\mathcal{P}$; the fields h and h_N, called heat fluxes and measured per unit length per unit time,[‡] assume the values $\bar{h}$ and $\bar{h}_N$ on $\partial\mathcal{R}$, respectively. The total external rate of supply of heat per unit mass of $\mathcal{J}$ is defined as

$$r + \sum_{N=1}^{K} r_N \quad . \tag{3.1}$$

Similarly, the total external rate of supply of heat per unit length per unit time across $\partial\mathcal{R}$ and the total internal curve flux of heat across $\partial\mathcal{P}$ per unit length per unit time are defined, respectively, by

$$-\bar{h} - \sum_{N=1}^{K} \bar{h}_N \quad \text{and} \quad -h - \sum_{N=1}^{K} h_N \quad . \tag{3.2}$$

We now define the ratios of the heat supplies r and r_N to temperatures θ and θ_N, respectively, as $s = s(\theta^\gamma, t)$ and $s_N = s_N(\theta^\gamma, t)$ and call these the <u>external rates of supply of entropy</u> per unit mass of $\mathcal{J}$. Further we define the ratios of $\bar{h}$ and $\bar{h}_N$ to the temperatures θ and θ_N, respectively, as the <u>external rates of curve supplies of entropy</u> $\bar{k}$ and $\bar{k}_N$ per unit length of $\partial\mathcal{R}$; and, similarly, we define the ratios of h and h_N to the temperatures θ and θ_N, respectively, as the <u>internal curve fluxes of entropy</u> $k = k(\theta^\gamma, t\,;\,\underset{\sim}{\nu})$ and $k_N = k_N(\theta^\gamma, t\,;\,\underset{\sim}{\nu})$ per unit length of $\partial\mathcal{P}$. The above definitions may conveniently be summarized by

$$\begin{gathered} s = r/\theta \;\;,\;\; s_N = r_N/\theta_N \;\;,\;\; \bar{k} = \bar{h}/\theta \;\;,\;\; \bar{k}_N = \bar{h}_N/\theta_N \;\;, \\ k = h/\theta \;\;,\;\; k_N = h_N/\theta_N \quad . \end{gathered} \tag{3.3}$$

We require that the fields $s_N, \bar{k}_N, k_N$, defined by $(3.3)_{2,4,6}$ all tend to finite limits as $\theta_N \to 0$ for each $N = 1, 2, \ldots, K$.

[‡]The sign convention for the internal curve fluxes of heat are such that these fields $-h$ and $-h_N$ represent fluxes entering $\mathcal{P}$ across the boundary curve $\partial\mathcal{P}$.

In addition to the thermal fields already introduced, throughout $\mathcal{S}$ we assume the existence of scalar fields $\eta = \eta(\theta^\alpha, t)$ and $\eta_N = \eta_N(\theta^\alpha, t)$, called <u>specific entropies</u> and <u>internal rates of production of entropies</u> $\xi = \xi(\theta^\gamma, t)$ and $\xi_N = \xi_N(\theta^\gamma, t)$ per unit mass of $\mathcal{S}$. The contributions of these internal rates of production of entropies to the internal rate of production of heat is

$$\theta\xi + \sum_{N=1}^{K} \theta_N \xi_N \tag{3.4}$$

per unit mass.

We now postulate balances of entropy for every material surface of $\mathcal{C}$ occupying a part $\mathcal{P}$ in the present configuration and write*

$$\frac{dH}{dt} = \int_{\mathcal{P}} \rho(s+\xi)d\sigma - \int_{\partial\mathcal{P}} k\, ds \quad , \quad H = \int_{\mathcal{P}} \rho\, \eta\, d\sigma \quad , \tag{3.5}$$

$$\frac{dH_N}{dt} = \int_{\mathcal{P}} \rho(s_N+\xi_N)d\sigma - \int_{\partial\mathcal{P}} k_N\, ds \quad (N=1,2,\ldots,K) \quad , \quad H_N = \int_{\mathcal{P}} \rho\, \eta_N\, d\sigma \quad . \tag{3.6}$$

By usual procedures, it can be shown from (3.5) and (3.6) that k and k_N are linear functions of $\underset{\sim}{\nu}$, i.e.,

$$\begin{gathered} k = \underset{\sim}{p} \cdot \underset{\sim}{\nu} = p^\alpha \nu_\alpha \quad , \quad k_N = \underset{\sim}{p}_N \cdot \underset{\sim}{\nu} = p_N^\alpha \nu_\alpha \quad , \\ \underset{\sim}{p} = p^\alpha \underset{\sim}{a}_\alpha \quad , \quad \underset{\sim}{p}_N = p_N^\alpha \underset{\sim}{a}_\alpha \quad , \end{gathered} \tag{3.7}$$

where $\underset{\sim}{p}, \underset{\sim}{p}_N$ are called <u>entropy flux vectors</u>. Then, from $(3.3)_{5,6}$ and $(3.7)_{1,2}$

$$h = \theta \underset{\sim}{p} \cdot \underset{\sim}{\nu} \quad , \quad h_N = \theta_N \underset{\sim}{p}_N \cdot \underset{\sim}{\nu} \quad \text{(no sum on N)} \tag{3.8}$$

and we may define heat flux vectors $\underset{\sim}{q}, \underset{\sim}{q}_N$ by

$$\begin{gathered} \underset{\sim}{q} = \theta \underset{\sim}{p} \quad , \quad \underset{\sim}{q}_N = \theta_N \underset{\sim}{p}_N \quad \text{(no sum on N)} \quad , \\ h = \underset{\sim}{q} \cdot \underset{\sim}{\nu} \quad , \quad h_N = \underset{\sim}{q}_N \cdot \underset{\sim}{\nu} \quad . \end{gathered} \tag{3.9}$$

*A motivation for postulating equations (3.5) and (3.6) for balances of entropy is provided by consideration of derivations from three-dimensional equations in §7.

Under suitable continuity assumptions and with the use of (3.7), the field equations resulting from (3.5) and (3.6) are

$$\rho\dot{\eta} = \rho(s + \xi) - \operatorname{div} \underset{\sim}{p} \quad ,$$
$$\rho\dot{\eta}_N = \rho(s_N + \xi_N) - \operatorname{div} \underset{\sim}{p}_N \quad , \tag{3.10}$$

where

$$\operatorname{div} \underset{\sim}{p} = a^{-\frac{1}{2}}(a^{\frac{1}{2}}p^{\alpha})_{,\alpha} \quad , \quad \operatorname{div} \underset{\sim}{p}_N = a^{-\frac{1}{2}}(a^{\frac{1}{2}}p_N^{\alpha})_{,\alpha} \quad .$$

We now introduce the first law of thermodynamics or the balance of energy for the Cosserat surface $\mathcal{C}$. This states that heat and mechanical energy are equivalent and that together they are conserved for every material surface of $\mathcal{C}$. Thus, with reference to the present configuration, the balance of energy may be stated in the form†

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}}\{\tfrac{1}{2}(\underset{\sim}{v}\cdot\underset{\sim}{v} + 2k^1\underset{\sim}{v}\cdot\underset{\sim}{w} + k^2\underset{\sim}{w}\cdot\underset{\sim}{w}) + \varepsilon\}\rho d\sigma \\ = \int_{\mathcal{P}}(r + \sum_{N=1}^{K} r_N + \underset{\sim}{f}\cdot\underset{\sim}{v} + \underset{\sim}{\ell}\cdot\underset{\sim}{w})\rho d\sigma \\ + \int_{\partial\mathcal{P}}(\underset{\sim}{N}\cdot\underset{\sim}{v} + \underset{\sim}{M}\cdot\underset{\sim}{w} - h - \sum_{N=1}^{K} h_N)ds \quad ,\end{aligned} \tag{3.12}$$

where $\varepsilon = \varepsilon(\theta^{\gamma},t)$ is the internal energy per unit mass of $\mathcal{S}$. With the help of (2.12) to (2.15), (3.9) and (3.10) and under suitable continuity assumptions, the field equation resulting from (3.12) is

$$\begin{aligned}-\rho(\dot{\varepsilon} - \theta\dot{\eta} - \sum_{N=1}^{K}\theta_N\dot{\eta}_N) - \rho(\theta\xi + \sum_{N=1}^{K}\theta_N\xi_N) - \underset{\sim}{p}\cdot\underset{\sim}{g} - \sum_{N=1}^{K}\underset{\sim}{p}_N\cdot\underset{\sim}{g}_N \\ + \underset{\sim}{N}^{\alpha}\cdot\underset{\sim}{v}_{,\alpha} + \underset{\sim}{m}\cdot\underset{\sim}{w} + \underset{\sim}{M}^{\alpha}\cdot\underset{\sim}{w}_{,\alpha} = 0 \quad ,\end{aligned} \tag{3.13}$$

where the temperature gradients $\underset{\sim}{g}$ and $\underset{\sim}{g}_N$ are defined by

---

†See Green and Naghdi (1977) for further remarks on this in the context of the three-dimensional theory.

$$\underset{\sim}{g} = \text{grad}\ \theta = \theta_{,\alpha}\underset{\sim}{a}^{\alpha}\ ,\quad \underset{\sim}{g}_N = \text{grad}\ \theta_N = \theta_{N,\alpha}\underset{\sim}{a}^{\alpha}\ . \tag{3.14}$$

Introducing the Helmholtz free energy $\psi = \psi(\theta^{\gamma},t)$ per unit mass of $\mathscr{S}$ by

$$\psi = \epsilon - \theta\eta - \sum_{N=1}^{K} \theta_N \eta_N\ , \tag{3.15}$$

the energy equation (3.15) may be written in the alternative form

$$-\rho(\dot{\psi} + \eta\dot{\theta} + \sum_{N=1}^{K} \eta_N\dot{\theta}_N) - \rho(\theta\xi + \sum_{N=1}^{K} \theta_N\xi_N) - \underset{\sim}{p}\cdot\underset{\sim}{g} - \sum_{N=1}^{K} \underset{\sim}{p}_N\cdot\underset{\sim}{g}_N$$

$$+ \underset{\sim}{N}^{\alpha}\cdot\underset{\sim}{v}_{,\alpha} + \underset{\sim}{m}\cdot\underset{\sim}{w} + \underset{\sim}{M}^{\alpha}\cdot\underset{\sim}{w}_{,\alpha} = 0\ . \tag{3.16}$$

For a given Cosserat surface having a reference density $\rho_0(\theta^{\alpha})$, the field equations obtained from the integral form of the conservation laws, involve a set of 5K+15 functions. These consist of the deformation functions $\underset{\sim}{r},\underset{\sim}{d}$ and the temperatures θ,θ_N, i.e.

$$\{\underset{\sim}{r},\underset{\sim}{d},\theta,\theta_N\}\ , \tag{3.17}$$

the various mechanical and the thermal fields, namely†

$$\{\underset{\sim}{N}^{\alpha},\underset{\sim}{M}^{\alpha},\underset{\sim}{m},\underset{\sim}{p},\underset{\sim}{p}_N,\psi,\eta,\eta_N,\xi,\xi_N\}\ , \tag{3.18}$$

and

$$\{\underset{\sim}{f},\underset{\sim}{\ell},s,s_N\}\ . \tag{3.19}$$

We assume that the fields (3.18) are specified by constitutive equations which may depend on the variables (3.17), their space and time derivatives, as well as the whole history of deformation and temperature. We then adopt the following procedure in utilizing the conservation laws:

†The mass density ρ is not included in (3.18) and (3.19) since, given (3.17), ρ can be calculated from (2.15).

(1) The field equations are assumed to hold for arbitrary choice of the functions (3.17) including, if required, an arbitrary choice of the space and time derivatives of these functions;

(2) The fields (3.18) are calculated from their respective constitutive equations;

(3) The values of the variables (3.19) can then be found from the balances of momenta (2.16) and (2.17) and balances of entropy (3.10);

(4) The equation (2.18) resulting from the balance of moment of momentum, and the equation (3.16) resulting from the energy equation, will be regarded as identities for every choice of (3.17). This will place restrictions on the constitutive equations.

We note that the quantities $\xi, \xi_N, \eta, \eta_N, \psi$ may be arbitrary to the extent of additive functions $\dot{f}, \dot{f}_N, f, f_N, -\theta f - \sum_{N=1}^{K} \theta_N f_N$, respectively, where f, f_N are arbitrary functions of the variables (3.17), their space and time derivatives and functionals of their histories. The additive functions have the property that they make no contribution to the differential equations for $\underset{\sim}{r}, \underset{\sim}{d}, \theta, \theta_N$ and the boundary and initial conditions. They also make no contribution to the energy identity (3.16) and no contribution to the internal energy ϵ. We remove this arbitrariness by setting[‡]

$$f = \hat{f}(\theta^{\gamma}) \quad , \quad f_N = \hat{f}_N(\theta^{\gamma}) \quad , \quad \dot{f} = 0 \quad , \quad \dot{f}_N = 0 \quad . \tag{3.20}$$

Then, the functions ξ, ξ_N are determined uniquely and η, η_N are only arbitrary to the extent of additive functions of θ^{γ}, independent of t. The functions $\hat{f}, \hat{f}_N$ in (3.20) can thus be determined by specifying values for η, η_N in some reference state.

So far no mention has been made of restrictions on constitutive equations which may arise from some form of second law of thermodynamics, usually

[‡]For a more elaborate parallel discussion in the context of the three-dimensional theory, see Green and Naghdi (1977, §2).

interpreted in terms of an "entropy inequality." Before considering this and in order to gain some insight into the nature of our procedure described above, we study in the next section the relatively simple case of an elastic shell.

For later use, we record the expressions for the external work and the external heat supplied to any part $\mathcal{P}$ of the surface $\mathcal{S}$ during the time interval $t_1 \leq t \leq t_2$. First, however, guided by the results of §4 we observe that in the case of an elastic Cosserat surface the response functions $\psi, \eta, \eta_N, \epsilon$ depend only on the vectors $\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}$ and the temperatures θ, θ_N $(N = 1,2,\ldots,\kappa)$ and are independent of their rates and the temperature gradients $\mathbf{g}, \mathbf{g}_N$. Such an elastic material will be regarded as nondissipative in a sense that will be made precise later; and in conjunction with an expression for the external mechanical work supplied to any part $\mathcal{P}$, will be used as a basis for establishing in §5 an inequality representing the second law of thermodynamics for dissipative materials. Keeping this background in mind, we assume that the constitutive response functions for ϵ, η include also dependence on the list of variables $\dot{\mathbf{a}}_\alpha, \dot{\mathbf{d}}, \dot{\mathbf{d}}_{,\beta}, \dot{\theta}, \dot{\theta}_N, \mathbf{g}, \mathbf{g}_N$ and their higher space and time derivatives and refer to this list collectively as $\mathcal{V}$. Further, let ϵ', η' denote the respective values of ϵ, η when the list $\mathcal{V}$ is put equal to zero in the response functions. Thus, for example,

$$
\begin{aligned}
\epsilon &= \epsilon(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \theta, \theta_N, \mathcal{V}) \ , \\
\epsilon' &= \epsilon'(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \theta, \theta_N) = \epsilon(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \theta, \theta_N, 0) \ , \qquad (3.21) \\
\mathcal{V} &= (\dot{\mathbf{a}}_\alpha, \dot{\mathbf{d}}, \dot{\mathbf{d}}_{,\gamma}, \dot{\theta}, \dot{\theta}_N, \mathbf{g}, \mathbf{g}_N, \ldots) \ ,
\end{aligned}
$$

where the dots in $(3.21)_3$ refer to the higher space and time derivatives of $\dot{\mathbf{a}}_\alpha, \dot{\mathbf{d}}, \dot{\mathbf{d}}_{,\beta}, \dot{\theta}, \dot{\theta}_N, \mathbf{g}, \mathbf{g}_N$. Then, with the help of (2.15) to (2.17) and the integral of (3.12) with respect to time, we obtain

$\mathfrak{W}$ = External mechanical work supplied to a part $\mathcal{P}$ of the shell during the time interval $t_1 \leqq t \leqq t_2$

$$= \int_{t_1}^{t_2}\Big[\int_{\mathcal{P}} (\underset{\sim}{f} \cdot \underset{\sim}{v} + \underset{\sim}{\ell} \cdot \underset{\sim}{w})\rho \, d\sigma + \int_{\partial\mathcal{P}} (\underset{\sim}{N} \cdot \underset{\sim}{v} + \underset{\sim}{M} \cdot \underset{\sim}{w}) ds\Big]dt$$

$$= \Delta K + \Delta E + \bar{\mathfrak{w}} + \mathfrak{w}_2 \tag{3.22}$$

and

$\mathfrak{H}$ = External heat supplied to a part $\mathcal{P}$ of the shell during the time interval $t_1 \leqq t \leqq t_2$

$$= \int_{t_1}^{t_2}\Big[\int_{\mathcal{P}} (r + \sum_{N=1}^{K} r_N)\rho \, d\sigma - \int_{\partial\mathcal{P}} (\underset{\sim}{q} + \sum_{N=1}^{K} \underset{\sim}{q}_N) \cdot \underset{\sim}{\nu} \, ds\Big]dt$$

$$= -\bar{\mathfrak{w}} - \mathfrak{w}_2 \ , \tag{3.23}$$

where

$$\bar{\mathfrak{w}} = -\int_{t_1}^{t_2}\int_{\mathcal{P}} \rho(\theta\dot{\eta}' + \sum_{N=1}^{K} \theta_N \dot{\eta}_N')d\sigma \, dt \ , \quad \mathfrak{w}_2 = \int_{t_1}^{t_2}\int_{\mathcal{P}} \rho \, w \, d\sigma \, dt \ , \tag{3.24}$$

K and E stand for the kinetic and internal energies defined by

$$K = \int_{\mathcal{P}} \tfrac{1}{2}(\underset{\sim}{v}^2 + 2k^1 \underset{\sim}{v} \cdot \underset{\sim}{w} + k^2 \underset{\sim}{w}^2)\rho \, d\sigma \ , \quad E = \int_{\mathcal{P}} \rho \, \epsilon \, d\sigma \ , \tag{3.25}$$

respectively, and where the prefix Δ denotes the difference operation on functions and fields during the time interval $[t_1, t_2]$, e.g., $\Delta E = E(t_2) - E(t_1)$. Also, w in $(3.24)_2$ is given by

$$\begin{aligned}\rho w &= -\rho(\dot{\epsilon} - \dot{\epsilon}') - \rho(\dot{\psi}' + \eta'\dot{\theta} + \sum_{N=1}^{K} \eta_N'\dot{\theta}_N) + \underset{\sim}{N}^{\alpha} \cdot \underset{\sim}{v}_{,\alpha} + \underset{\sim}{m} \cdot \underset{\sim}{w} + \underset{\sim}{M}^{\alpha} \cdot \underset{\sim}{w}_{,\alpha} \\ &= -\rho[(\dot{\eta} - \dot{\eta}')\theta + \sum_{N=1}^{K} (\dot{\eta}_N - \dot{\eta}_N')\theta_N] \\ &\quad + \rho(\theta\xi + \sum_{N=1}^{K} \theta_N \xi_N) + \underset{\sim}{p} \cdot \underset{\sim}{g} + \sum_{N=1}^{K} \underset{\sim}{p}_N \cdot \underset{\sim}{g}_N \ ,\end{aligned} \tag{3.26}$$

$$\psi' = \epsilon' - \theta\eta' - \sum_{N=1}^{K} \theta_N \eta_N' \ . \tag{3.27}$$

The foregoing discussion of thermodynamics of shells includes, of course, the thermodynamics of a membrane surface or an interface. Results of this kind can be recovered by deleting the terms which involve the director.

4. Thermoelastic theory of a Cosserat surface.

Previous work on the thermoelastic theory of a Cosserat surface made use of a two dimensional Clausius-Duhem inequality and only one temperature field was considered, which corresponds to the surface temperature θ of the present paper (see Naghdi 1972). We consider now constitutive equations for a thermoelastic Cosserat surface which admits K+1 temperature fields and we examine the restrictions imposed on these equations by the procedure described at the end of §3.

We assume that the set of variables

$$\mathbf{N}^{\alpha}, \mathbf{M}^{\alpha}, \mathbf{m}, \mathbf{p}, \mathbf{p}_N, \psi, \eta, \eta_N, \xi, \xi_N \tag{4.1}$$

are functions of the set

$$\mathbf{a}_{\alpha}, \mathbf{d}, \mathbf{d}_{,\gamma}, \theta, \theta_N, \mathbf{g}, \mathbf{g}_N \ , \tag{4.2}$$

as well as the reference values

$$\mathbf{A}_{\alpha}, \mathbf{D}, \mathbf{D}_{,\gamma}, \Theta \ , \tag{4.3}$$

and in addition may depend also on the particle θ^{μ}. In the set of reference values (4.3), Θ is the constant reference value of θ and we have assumed the reference values of θ_N $(N=1,2,\ldots,K)$ to be zero. Postponing the restrictions to be imposed by the invariance requirements under superposed rigid-body motions and recalling the procedure outlined in §3, the energy equation (3.16) is identically satisfied for all thermo-mechanical processes provided

$$\partial\psi/\partial\mathbf{g} = 0 \quad , \quad \partial\psi/\partial\mathbf{g}_N = 0 \tag{4.4}$$

and

$$\psi = \hat{\psi}(\underset{\sim}{a}_\alpha, \underset{\sim}{d}, \underset{\sim}{d}_{,\gamma}, \theta, \theta_N ; \underset{\sim}{A}_\alpha, \underset{\sim}{D}, \underset{\sim}{D}_{,\gamma}, \Theta ; \theta^\mu) \quad , \tag{4.5}$$

$$\underset{\sim}{N}^\alpha = \rho \frac{\partial\hat{\psi}}{\partial \underset{\sim}{a}_\alpha} \quad , \quad \underset{\sim}{M}^\alpha = \rho \frac{\partial\hat{\psi}}{\partial \underset{\sim}{d}_{,\alpha}} \quad , \quad \underset{\sim}{m} = \rho \frac{\partial\hat{\psi}}{\partial \underset{\sim}{d}} \quad , \tag{4.6}$$

$$\eta = -\frac{\partial\hat{\psi}}{\partial\theta} \quad , \quad \eta_N = -\frac{\partial\hat{\psi}}{\partial\theta_N} \quad , \tag{4.7}$$

$$\rho(\theta\xi + \sum_{N=1}^{K} \theta_N \xi_N) + \underset{\sim}{p} \cdot \underset{\sim}{g} + \sum_{N=1}^{K} \underset{\sim}{p}_N \cdot \underset{\sim}{g}_N = 0 \quad , \tag{4.8}$$

where as indicated in (4.5) the function $\hat{\psi}$ is independent of the temperature gradients $\underset{\sim}{g}$ and $\underset{\sim}{g}_N$. Formulae (4.6) and $(4.7)_1$, with θ_N absent from (4.5), were obtained by Naghdi (1972, Sec. 13) with the help of the Clausius-Duhem inequality and he also showed how to obtain alternative forms with the help of invariance conditions under superposed rigid body motions. In this connection, it may be recalled that in the discussion of some forms of the constitutive equations for an elastic Cosserat surface, Naghdi (1972, Sec. 13) at first indicated the dependence of the response functions on the properties of a physically preferred reference state and material inhomogeneity through the argument θ^μ and subsequently specified a more explicit dependence on the reference values such as (4.3) in order to obtain other forms of constitutive equations in terms of relative kinematic measures.* Results for an elastic membrane or an interface (regarded as a membrane) follow from (4.5) to (4.8) by suppressing the directors $\underset{\sim}{D}, \underset{\sim}{d}$ and omitting the response functions for $\underset{\sim}{M}^\alpha, \underset{\sim}{m}$.

It is now convenient to introduce the component form of the kinematic variables $\underset{\sim}{d}, \underset{\sim}{d}_{,\alpha}$ relative to the base vectors $\underset{\sim}{a}_i$ or $\underset{\sim}{a}^i$, and the component form of $\underset{\sim}{D}, \underset{\sim}{D}_{,\alpha}$ relative to $\underset{\sim}{A}_i$ or $\underset{\sim}{A}^i$. Thus, we write

---

*The influence of the reference geometry on the response of elastic shells has been examined also by Carroll and Naghdi (1972) who assumed the existence of a local preferred state of the body and then stipulated that the influence of the reference geometry, as in (4.5), occurs through the values of the constitutive variables in the preferred state.

$$\underset{\sim}{d} = d_i \underset{\sim}{a}^i = d^i \underset{\sim}{a}_i \ , \quad \underset{\sim}{d}_{,\alpha} = \lambda_{i\alpha} \underset{\sim}{a}^i = \lambda^i_{\cdot\alpha} \underset{\sim}{a}_i \ ,$$

$$\underset{\sim}{D} = D_i \underset{\sim}{A}^i = D^i \underset{\sim}{A}_i \ , \quad \underset{\sim}{D}_{,\alpha} = \Lambda_{i\alpha} \underset{\sim}{A}^i = \Lambda^i_{\cdot\alpha} \underset{\sim}{A}_i \ , \tag{4.9}$$

where

$$d_i = \underset{\sim}{d} \cdot \underset{\sim}{a}_i \ , \quad d^i = a^{ij} d_j \ , \quad \lambda_{i\alpha} = \underset{\sim}{d}_{,\alpha} \cdot \underset{\sim}{a}_i \ , \quad \lambda^i_{\cdot\alpha} = a^{ij} \lambda_{j\alpha} \ ,$$

$$D_i = \underset{\sim}{D} \cdot \underset{\sim}{A}_i \ , \quad D^i = A^{ij} D_j \ , \quad \Lambda_{i\alpha} = \underset{\sim}{D}_{,\alpha} \cdot \underset{\sim}{A}_i \ , \quad \Lambda^i_{\cdot\alpha} = A^{ij} \Lambda_{j\alpha} \ , \tag{4.10}$$

$$a^{i3} = a^{3i} = 0 \ , \quad a^{33} = 1 \ , \quad A^{i3} = A^{3i} = 0 \ , \quad A^{33} = 1$$

and

$$\lambda_{\alpha\beta} = d_{\alpha|\beta} - b_{\alpha\beta} d_3 \ , \quad \lambda_{3\alpha} = d_{3,\alpha} + b^\beta_\alpha d_\beta \ ,$$

$$\Lambda_{\alpha\beta} = D_{\alpha\|\beta} - B_{\alpha\beta} D_3 \ , \quad \Lambda_{3\alpha} = D_{3,\alpha} + B^\beta_\alpha D_\beta \ . \tag{4.11}$$

In the above formulae, a single vertical bar stands for covariant differentiation using Christoffel symbols formed from $a_{\alpha\beta}$, a double vertical line denotes covariant differentiation using Christoffel symbols formed from $A_{\alpha\beta}$ and

$$b_{\alpha\beta} = \underset{\sim}{a}_3 \cdot \underset{\sim}{a}_{\alpha,\beta} \ , \quad B_{\alpha\beta} = \underset{\sim}{A}_3 \cdot \underset{\sim}{A}_{\alpha,\beta} \tag{4.12}$$

are the coefficients of the second fundamental form of the surfaces $\mathscr{s}$ and $\mathscr{S}$, respectively.

With the help of invariance conditions under superposed rigid body motions the constitutive equations (4.5) to (4.7) may be expressed in an alternative form which will be utilized later in the paper.[†] Under such motions the vectors $\underset{\sim}{a}_\alpha, \underset{\sim}{d}, \underset{\sim}{d}_{,\gamma}$ become $\underset{\sim}{Q}\,\underset{\sim}{a}_\alpha$, $\underset{\sim}{Q}\,\underset{\sim}{d}$, $\underset{\sim}{Q}\,\underset{\sim}{d}_{,\gamma}$, where $\underset{\sim}{Q}$ is a proper orthogonal function of the time, and the value ψ^+ of the response function $\hat{\psi}$ in (4.5) is given by

[†]See Naghdi (1972, Sect. 13).

$$\psi^+ = \hat{\psi}(\underset{\sim}{Q}\,\underset{\sim\alpha}{a},\ \underset{\sim}{Q}\,\underset{\sim}{d},\ \underset{\sim}{Q}\,\underset{\sim,\gamma}{d},\ \theta,\ \theta_N,\ \underset{\sim\alpha}{A},\ \underset{\sim}{D},\ \underset{\sim,\gamma}{D},\ \Theta\,;\,\theta^\mu) \quad . \tag{4.13}$$

We assume that the value of the free energy is unaltered by superposed rigid body motions so that $\psi^+ = \psi$ or

$$\hat{\psi}(\underset{\sim}{Q}\,\underset{\sim\alpha}{a},\ \underset{\sim}{Q}\,\underset{\sim}{d},\ \underset{\sim}{Q}\,\underset{\sim,\gamma}{d},\ldots) = \hat{\psi}(\underset{\sim\alpha}{a},\underset{\sim}{d},\underset{\sim,\gamma}{d},\ldots) \tag{4.14}$$

for all proper orthogonal tensors $\underset{\sim}{Q}$. At this point it should be mentioned that those who prefer the concept of objectivity will make the statement (4.14) for the full orthogonal group.

Assuming first that (4.14) holds only for the proper orthogonal group for $\underset{\sim}{Q}$ then from Cauchy's representation theorem $\hat{\psi}$ may be expressed as a different function of the inner products and scalar triple products of $\underset{\sim\alpha}{a},\underset{\sim}{d},\underset{\sim,\gamma}{d}$, namely

$$\begin{aligned}
&\underset{\sim\alpha}{a}\cdot\underset{\sim\beta}{a} = a_{\alpha\beta}\ ,\quad \underset{\sim\alpha}{a}\cdot\underset{\sim}{d} = d_\alpha\ ,\quad \underset{\sim\beta}{a}\cdot\underset{\sim,\alpha}{d} = \lambda_{\beta\alpha}\ ,\\
&\underset{\sim}{d}\cdot\underset{\sim}{d} = d^i d_i\ ,\quad \underset{\sim}{d}\cdot\underset{\sim,\alpha}{d} = d^i\lambda_{i\alpha}\ ,\quad \underset{\sim,\alpha}{d}\cdot\underset{\sim,\beta}{d} = \lambda^i_{\cdot\alpha}\lambda_{i\beta}
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&[\underset{\sim1}{a}\underset{\sim2}{a}\underset{\sim}{d}] = a^{\frac{1}{2}} d_3\ ,\quad [\underset{\sim1}{a}\underset{\sim2}{a}\underset{\sim,\alpha}{d}] = a^{\frac{1}{2}}\lambda_{3\alpha}\ ,\\
&[\underset{\sim\alpha}{a}\underset{\sim}{d}\underset{\sim,\beta}{d}] = d^3\epsilon_{\gamma\alpha}\lambda^\gamma_{\cdot\beta} + \lambda^3_{\cdot\beta}\epsilon_{\alpha\gamma}d^\gamma\ .
\end{aligned} \tag{4.16}$$

In view of $(2.2)_5$, $[\underset{\sim1}{a}\underset{\sim2}{a}\underset{\sim3}{a}] = a^{\frac{1}{2}} > 0$, so that $a^{\frac{1}{2}}$ can be expressed as a single-valued function of $a_{\alpha\beta}$. It then follows from (4.15) and (4.16) that $\hat{\psi}$ may be expressed as a single-valued function of $a_{\alpha\beta}, d_i, \lambda_{i\alpha}$, the remaining contributions in (4.16) being redundant. Similarly, if we make the other choice of the normal $\underset{\sim3}{a}$ in (2.2) so that $[\underset{\sim1}{a}\underset{\sim2}{a}\underset{\sim3}{a}] = -\,a^{\frac{1}{2}}$, we may again reduce $\hat{\psi}$ to be a single-valued function of $a_{\alpha\beta}, d_i, \lambda_{i\alpha}$. Moreover, when $[\underset{\sim1}{A}\underset{\sim2}{A}\underset{\sim3}{A}] > 0$ or < 0, we may express $\underset{\sim3}{A}$ as a single-valued function of $\underset{\sim\alpha}{A}$, and replace the dependence of $\hat{\psi}$ on $\underset{\sim\alpha}{A},\underset{\sim}{D},\underset{\sim,\beta}{D}$ by

$\underset{\sim}{A}_\alpha, D_i, \Lambda_{i\alpha}$. Hence (4.5) is replaced by the different single-valued function†

$$\psi = \tilde{\psi}(a_{\alpha\beta}, d_i, \lambda_{i\alpha}, \theta, \theta_N ; \underset{\sim}{A}_\alpha, D_i, \Lambda_{k\alpha}, \Theta ; \theta^\mu) \quad . \tag{4.17}$$

Let $N^{\alpha i}, M^{\alpha i}, m^i$ denote, respectively, the components of $\underset{\sim}{N}^\alpha, \underset{\sim}{M}^\alpha$ and $\underset{\sim}{m}$ referred to the base vectors $\underset{\sim}{a}_i$, i.e.,

$$\underset{\sim}{N}^\alpha = N^{\alpha i}\underset{\sim}{a}_i \quad , \quad \underset{\sim}{M}^\alpha = M^{\alpha i}\underset{\sim}{a}_i \quad , \quad \underset{\sim}{m} = m^i \underset{\sim}{a}_i \quad . \tag{4.18}$$

Then, with the help of (2.2) and (4.10), it follows from (4.5), (4.6), (4.7) and (4.17) that

$$\begin{gathered}
m^i = \rho \frac{\partial\tilde{\psi}}{\partial d_i} \quad , \quad M^{\alpha i} = \rho \frac{\partial\tilde{\psi}}{\partial \lambda_{i\alpha}} \quad , \\
N'^{\alpha\beta} = N^{\alpha\beta} - m^\alpha d^\beta - M^{\gamma\alpha}\lambda^\beta_{\cdot\gamma} = \rho\left(\frac{\partial\tilde{\psi}}{\partial a_{\alpha\beta}} + \frac{\partial\tilde{\psi}}{\partial a_{\beta\alpha}}\right) \quad , \\
N^{\alpha 3} - m^\alpha d^3 - M^{\gamma\alpha}\lambda_{3\gamma} + m^3 d^\alpha + M^{\gamma 3}\lambda^\alpha_{\cdot\gamma} = 0 \quad , \\
\eta = -\frac{\partial\tilde{\psi}}{\partial\theta} \quad , \quad \eta_N = -\frac{\partial\tilde{\psi}}{\partial\theta_N} \quad .
\end{gathered} \tag{4.19}$$

Apart from the last formula in (4.19), the above results are equivalent to those given by Green, Naghdi and Wainwright (1965). The moment of momentum equation (2.18) is satisfied by $(4.19)_{3,4}$.

In order to complete the discussion of invariance under superposed rigid body motions it remains to consider $\underset{\sim}{p}, \underset{\sim}{p}_N, \xi, \xi_N$. For example, recalling $(3.7)_3$, we have the constitutive relation

$$p^\alpha = \hat{p}^\alpha(\underset{\sim}{a}_\beta, \underset{\sim}{d}, \underset{\sim}{d}_{,\gamma}, \theta, \theta_N, \underset{\sim}{g}, \underset{\sim}{g}_N ; \underset{\sim}{A}_\beta, \underset{\sim}{D}, \underset{\sim}{D}_{,\gamma}, \Theta ; \theta^\mu)$$

subject to the condition

† The response function $\tilde{\psi}$ in (4.17) exhibits explicitly dependence on the basis $\underset{\sim}{A}_\alpha$ in the reference configuration of the surface of $\mathcal{C}$. Since $\tilde{\psi}$ is a scalar-valued function, its dependence on $\underset{\sim}{A}_\alpha$ will be through $A_{\alpha\beta} = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{A}_\beta$.

$$\hat{p}^{\alpha}(\underset{\sim}{Q}\,\underset{\sim}{a}_{\beta},\ \underset{\sim}{Q}\,\underset{\sim}{d},\ \underset{\sim}{Q}\,\underset{\sim}{d}_{,\gamma},\ \theta,\ \theta_N,\ \underset{\sim}{Q}\,\underset{\sim}{g},\ \underset{\sim}{Q}\,\underset{\sim}{g}_N\ ;\ \ldots)$$

$$= \hat{p}^{\alpha}(\underset{\sim}{a}_{\beta},\underset{\sim}{d},\underset{\sim}{d}_{,\gamma},\theta,\theta_N,\underset{\sim}{g},\underset{\sim}{g}_N\ ;\ \ldots)$$

for all proper orthogonal tensors $\underset{\sim}{Q}$. By a discussion similar to that used for $\tilde{\psi}$, it follows that p^{α} can be expressed in the different functional form

$$p^{\alpha} = \hat{p}^{\alpha}(a_{\alpha\beta},d_i,\lambda_{i\alpha},\theta,\theta_N,\ \frac{\partial\theta}{\partial\theta^{\mu}},\ \frac{\partial\theta_N}{\partial\theta^{\mu}}\,;\,\underset{\sim}{A}_{\alpha},D_i,\Lambda_{k\gamma},\Theta\,;\,\theta^{\mu}) \quad .$$

Similar results follow for p_N^{α},ξ,ξ_N.

Although we adopt the representation (4.17) in the rest of this paper, we note that if (4.13) is to hold for the full orthogonal group, then $\tilde{\psi}$ becomes a function of the inner products (4.15). Moreover, since $\underset{\sim}{a}_3 \cdot \underset{\sim}{d} = d_3 \neq 0$, we may reject redundant elements in (4.15), and reduce $\hat{\psi}$ to a function of

$$a_{\alpha\beta},d_{\alpha},\lambda_{\alpha\beta},(d_3)^2,d^3\lambda_{3\alpha}\,;\,\underset{\sim}{A}_{\alpha},D_i,\Lambda_{k\alpha},\Theta\,;\,\theta^{\mu}$$

as found by Naghdi (1972, Eq. (13.36)). On the other hand, the component d_3 (since it is nonzero) can either be chosen to be always >0 or always <0. In each case d_3 may be expressed as a single-valued function of $(d_3)^2$ so that ψ as a function of the variables (4.20a) can be expressed in the form (4.17) and conversely. This means that there is no difference in results if we use the orthogonal group instead of the proper orthogonal group for our invariance conditions. Moreover, there are different, but equivalent, representations for ψ.

Returning to our main objective, with the help of (4.8) and (3.22)-(3.24), for an elastic Cosserat surface we record below the expressions representing (i) work by assigned force and assigned director force and by contact force and contact director force acting on any part $\mathcal{P}$ and (ii) supply of energy arising from the total external rate of supply of heat and the total curve flux of heat to $\mathcal{P}$, both over a finite time interval $t_1 \leq t \leq t_2$:

$$
\begin{aligned}
\mathfrak{W} &= \int_{t_1}^{t_2} \Big[\int_{\mathcal{P}} (\underset{\sim}{f} \cdot \underset{\sim}{v} + \underset{\sim}{\ell} \cdot \underset{\sim}{w}) \rho \, d\sigma + \int_{\partial \mathcal{P}} (\underset{\sim}{N} \cdot \underset{\sim}{v} + \underset{\sim}{M} \cdot \underset{\sim}{w}) ds \Big] dt \\
&= \Delta K + \Delta E' + \bar{\mathfrak{W}}
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
\mathfrak{H} &= \int_{t_1}^{t_2} \Big[\int_{\mathcal{P}} (r + \sum_{N=1}^{K} r_N) \rho \, d\sigma - \int_{\partial \mathcal{P}} (\underset{\sim}{q} + \sum_{N=1}^{K} \underset{\sim}{q}_N) \cdot \underset{\sim}{\nu} \, ds \Big] dt \\
&= \bar{\mathfrak{W}} ,
\end{aligned}
\tag{4.22}
$$

where $\bar{\mathfrak{W}}$ and K are given by $(3.24)_1$ and $(3.25)_1$, respectively, and E' is defined by

$$
E' = \int_{\mathcal{P}} \rho \epsilon' d\sigma \ . \tag{4.23}
$$

5. The second law of thermodynamics.

Previously (Green and Naghdi 1977,1978), in the context of the three-dimensional theory we have discussed the nature of thermodynamic irreversibility arising from a mathematical interpretation of a statement of the second law of thermodynamics that "it is impossible completely to reverse a process in which energy is transformed into heat by friction." Here we follow the same procedure and reconsider a mathematical interpretation of a second law appropriate for a Cosserat surface which admits more than one temperature field. Earlier work on the subject made use of a Clausius-Duhem inequality when only one temperature field is admitted (Green, Naghdi and Wainwright 1965, Naghdi 1972).

For the sake of clarity, we first dispose of some definitions. A <u>state of the Cosserat surface $\mathcal{C}$</u> at time t, regarded as representing a thin shell-like body, is described by the position vector $\mathbf{r}$ and the director $\mathbf{d}$, the velocities $(2.4)_{1,2}$, the temperatures θ and θ_N $(N=1,2,\ldots,K)$ throughout the surface $\mathscr{S}$ of $\mathcal{C}$, together with the constitutive response functions for the fields (3.18). Once the response functions are given, we then know the values of H and H_N in $(3.5)_2$ and $(3.6)_2$, as well as K and E in $(3.25)_{1,2}$. A <u>thermo-mechanical process</u> or simply a <u>process</u> is a time sequence of states; it is a continuous oriented curve in the space of states, i.e., the $(\theta,\theta_N,a_{\alpha\beta},d_i,\lambda_{i\alpha})$-space. Thus, a process may be defined by a sequence of values of

$$(\theta,\theta_N,\mathbf{r},\mathbf{d}) \tag{5.1}$$

throughout $\mathscr{S}$ in the time interval $0 \leqq t \leqq \sigma$. Similarly, the <u>reverse process</u> is a process defined by a sequence of values of (5.1) throughout $\mathscr{S}$ in the time interval $\sigma \leqq t \leqq 2\sigma$ subject to the conditions

$$\begin{aligned} &\theta(t) = \theta(2\sigma - t) \quad , \quad \theta_N(t) = \theta_N(2\sigma - t) \quad , \\ &\mathbf{r}(t) = \mathbf{r}(2\sigma - t) \quad , \quad \mathbf{d}(t) = \mathbf{d}_N(2\sigma - t) \quad . \end{aligned} \tag{5.2}$$

Returning to our main objective, we observe that in any process the work done by the external mechanical forces acting on $\mathcal{P}$ is positive or negative depending on whether the external work is supplied to, or is withdrawn from, $\mathcal{P}$. In general, some of the work done results in a change of the kinetic and internal energies represented by the first two terms on the right-hand side of $(3.22)_3$, each of which may be positive, negative or zero. Also, part of the work done may be positive with a corresponding extraction from $\mathcal{P}$ as heat or negative with a corresponding absorption of heat by $\mathcal{P}$. We note that in the case of an elastic material, the different contributions to $\mathfrak{w}$ will vary in sign depending on the process and will not be restricted to be either positive or negative for all processes. Consider any smooth process in the time interval $0 \leqq t \leqq \sigma$ and its reverse process in the time interval $\sigma \leqq t \leqq 2\sigma$. If the process is reversed in the $(\theta, \theta_N, a_{\alpha\beta}, d_i, \lambda_{i\alpha})$-space in such a way that at the end of the process and its reverse process the shell has returned to its original state with $\Delta\theta = 0$, $\Delta\theta_N = 0$, $\Delta a_{\alpha\beta} = 0$, $\Delta d_i = 0$, $\Delta\lambda_{i\alpha} = 0$, $\Delta\underset{\sim}{v} = \underset{\sim}{0}$, $\Delta\underset{\sim}{w} = \underset{\sim}{0}$ and, hence, $\Delta\epsilon = 0$, $\Delta\eta = 0$, $\Delta\eta_N = 0$, $\Delta\xi = 0$, $\Delta\xi_N = 0$, $\Delta\underset{\sim}{N}^{\alpha} = \underset{\sim}{0}$, $\Delta\underset{\sim}{M}^{\alpha} = \underset{\sim}{0}$, $\Delta\underset{\sim}{m} = \underset{\sim}{0}$, $\Delta\underset{\sim}{p} = \underset{\sim}{0}$, $\Delta\underset{\sim}{p}_N = \underset{\sim}{0}$ and $\Delta K = 0$, $\Delta E = 0$, then all the work done in the process is recovered as work in the reverse process.† This recovery of work would not be possible if in every arbitrary process part of $\mathfrak{w}$ always has a positive sign. With this motivation in mind, we assume that for any arbitrary process in a dissipative shell only part of the work done is recoverable as work in a reverse process, the rest being transformed into heat. We therefore assume that in every process part of the work done is always <u>nonnegative</u>. Then, if at the end of any process and its reverse process the shell has returned to the same state, some of the work done is always transformed into heat. Recalling that $\mathfrak{w}_2 = 0$ in $(3.24)_2$ in the case of an elastic shell for all processes, we interpret the above assumption for a dissipative

†If work is extracted in the process, then it is absorbed by the medium in the reverse process.

shell by requiring that

$$\mathfrak{w}_2 \geqq 0 \tag{5.3}$$

for all parts $\mathcal{P}$ and all processes, where $\mathfrak{w}_2$ is given by $(3.24)_2$. Since t_1, t_2 are arbitrary and ρw has already been assumed to be continuous, it follows that

$$\rho w = -\rho(\dot{\epsilon}-\dot{\epsilon}') - \rho(\dot{\psi}' + \eta'\dot{\theta} + \sum_{N=1}^{K} \eta'\dot{\theta}_N) + \underset{\sim}{N} \cdot \underset{\sim}{v}_{,\alpha} + \underset{\sim}{m} \cdot \underset{\sim}{w} + \underset{\sim}{M}^{\alpha} \cdot \underset{\sim}{w}_{,\alpha} \geqq 0 \tag{5.4}$$

for all thermo-mechanical processes. Also, from (3.23) and (5.4), we have

$$\mathcal{H} \leqq \int_{t_1}^{t_2}\int_{\mathcal{P}} \rho(\theta\dot{\eta}' + \sum_{N=1}^{K} \theta_N \dot{\eta}_N') d\sigma \, dt \tag{5.5}$$

so that the external heat supplied to a part $\mathcal{P}$ of the surface of $\mathcal{C}$ is bounded above in any process.

6. Summary of results from three-dimensional mechanical theory.

Consider a three-dimensional body, embedded in a Euclidean 3-space, and let its particles be identified by a convected coordinate θ^i $(i=1,2,3)$. Let $\underset{\sim}{r}^*$ denote the position vector, relative to a fixed origin, of a typical particle of the three-dimensional body in the present configuration at time t. Then,

$$\underset{\sim}{r}^* = \underset{\sim}{r}^*(\theta^1,\theta^2,\theta^3,t) \ , \quad \underset{\sim}{g}_i = \partial\underset{\sim}{r}^*/\partial\theta^i \ , \quad \underset{\sim}{v}^* = \dot{\underset{\sim}{r}}^* \ ,$$
$$g_{ik} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_k \ , \quad \underset{\sim}{g}^i \cdot \underset{\sim}{g}_k = \delta^i_k \ , \quad g^{ik} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^k \ , \quad g = \det g_{ik} \ , \tag{6.1}$$

where $\underset{\sim}{g}_i$ and $\underset{\sim}{g}^i$ are covariant and contravariant base vectors, respectively, g_{ik} and g^{ik} are covariant and contravariant metric tensors, respectively, and δ^i_k is the Kronecker delta. Also, a superposed dot denotes material time derivative holding θ^i fixed and $\underset{\sim}{v}^*$ is the velocity vector.

The stress vector $\underset{\sim}{t}$ across a surface in the present configuration whose unit outward normal is $\underset{\sim}{n}$ is given by

$$\underset{\sim}{t} = n_i \underset{\sim}{T}^i g^{\frac{1}{2}} = n_i \tau^{ik} \underset{\sim}{g}_k \ , \quad \underset{\sim}{n} = n_i \underset{\sim}{g}^i = n^i \underset{\sim}{g}_i \ , \tag{6.2}$$

where τ^{ik} are the contravariant components of the symmetric stress tensor. We do not recall here the consequences of the conservation laws of the three-dimensional theory since they will not be used explicitly in the present paper.

The parametric equation $\theta^3 = 0$ defines a surface in space at time t, which we assume to be smooth and non-intersecting. Any point of this surface is specified by the position vector $\underset{\sim}{r}$, relative to the same fixed origin to which $\underset{\sim}{r}^*$ is referred, where†

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^1,\theta^2,t) = \underset{\sim}{r}^*(\theta^1,\theta^2,0,t) \ . \tag{6.3}$$

Let the boundary of the three-dimensional continuum be specified by the material surfaces

† For convenience, we adopt the notation for $\underset{\sim}{r}$ in $(2.3)_1$ also for the surface (6.3). This permits an easy identification of the two surfaces.

$$\theta^3 = \alpha(\theta^1,\theta^2) \quad , \quad \theta^3 = \beta(\theta^1,\theta^2) \quad , \quad (\alpha < 0 < \beta) \quad , \tag{6.4}$$

with the surface $\theta^3 = 0$ lying entirely between $(6.4)_{1,2}$ and a material surface

$$f(\theta^1,\theta^2) = 0 \tag{6.5}$$

which is such that $\theta^3 =$ const. are closed smooth curves on the surface (6.5). Since later we identify the surface described by (6.3) with the surface $(2.3)_1$, it is convenient to adopt the notations for $\underset{\sim}{a}_\alpha, a_{\alpha\beta}$, etc. defined by (2.1) and (2.2) also for the surface (6.3).

We may now consider a general representation for $\underset{\sim}{r}^*$ in $(6.1)_1$ as a polynomial in θ^3 but in what follows we restriction attention to the approximation

$$\underset{\sim}{r}^* = \underset{\sim}{r} + \theta^3 \underset{\sim}{d} \quad , \quad \underset{\sim}{d} = \underset{\sim}{d}(\theta^\alpha, t) \tag{6.6}$$

in the bounded region $\alpha \leq \theta^3 \leq \beta$. Recalling (6.1), the velocity vector is then given by

$$\underset{\sim}{v}^* = \underset{\sim}{v} + \theta^3 \underset{\sim}{w} \quad , \quad \underset{\sim}{w} = \dot{\underset{\sim}{d}} \quad . \tag{6.7}$$

Given the approximation $(6.6)_1$, it is known (see, e.g., Naghdi 1972; Secs. 11, 12) that the field equations of the forms (2.15) to (2.18) can be derived from the three-dimensional field equations provided we identify $\underset{\sim}{d}$ in (6.6) with $(2.3)_2$ and adopt the definitions

$$\rho a^{\frac{1}{2}} = \lambda = \int_\alpha^\beta \rho^* g^{\frac{1}{2}} d\theta^3 \quad , \quad \lambda k^N = \int_\alpha^\beta \rho^* g^{\frac{1}{2}} (\theta^3)^N d\theta^3 \quad , \tag{6.8}$$

$$\underset{\sim}{N}^\alpha a^{\frac{1}{2}} = \int_\alpha^\beta \underset{\sim}{T}^\alpha d\theta^3 \quad , \quad \underset{\sim}{M}^\alpha a^{\frac{1}{2}} = \int_\alpha^\beta \underset{\sim}{T}^\alpha \theta^3 d\theta^3 \quad , \quad \underset{\sim}{m} a^{\frac{1}{2}} = \int_\alpha^\beta \underset{\sim}{T}^3 d\theta^3 \quad , \tag{6.9}$$

where ρ^* is the three-dimensional mass density. Also the assigned force $\underset{\sim}{f}$ and the assigned director force $\underset{\sim}{\ell}$ are related to the three-dimensional body force f^* per unit mass and to the effects of the stress vector $(6.2)_1$ over the boundary surfaces $(6.4)_{1,2}$ by

$$\lambda \underset{\sim}{f} = \int_{\alpha}^{\beta} \rho^* \underset{\sim}{f}^* g^{\frac{1}{2}} d\theta^3 + [\underset{\sim}{t} g^{\frac{1}{2}} \bar{\gamma}_{\alpha}(\theta^3)]_{\theta^3=\alpha} + [\underset{\sim}{t} g^{\frac{1}{2}} \bar{\gamma}_{\beta}(\theta^3)]_{\theta^3=\beta} \quad , \tag{6.10}$$

$$\lambda \underset{\sim}{\ell} = \int_{\alpha}^{\beta} \rho^* \underset{\sim}{f}^* g^{\frac{1}{2}} \theta^3 d\theta^3 + [\underset{\sim}{t} g^{\frac{1}{2}} \theta^3 \bar{\gamma}_{\alpha}(\theta^3)]_{\theta^3=\alpha} + [\underset{\sim}{t} g^{\frac{1}{2}} \theta^3 \bar{\gamma}_{\beta}(\theta^3)]_{\theta^3=\beta} \quad , \tag{6.11}$$

where

$$\bar{\gamma}_{\alpha}(\theta^3) = [g^{11}(\frac{\partial\alpha}{\partial\theta^1})^2 + g^{22}(\frac{\partial\alpha}{\partial\theta^2})^2 + g^{33}$$

$$+ 2(g^{12}\frac{\partial\alpha}{\partial\theta^1}\frac{\partial\alpha}{\partial\theta^2} - g^{13}\frac{\partial\alpha}{\partial\theta^1} - g^{23}\frac{\partial\alpha}{\partial\theta^2})]^{\frac{1}{2}} \tag{6.12}$$

and $\bar{\gamma}_{\beta}(\theta^3)$ is obtained from (6.9) by replacing α by β.

7. Thermodynamical results from three-dimensional theory.

In this section we obtain some thermodynamical results for a shell-like body on the basis of the recent thermodynamical theory of Green and Naghdi (1977). Thus, along with the (three-dimensional) temperature field‡ $\theta^* = \theta^*(\theta^i,t) > 0$, we admit the existence of an external rate of supply of heat $-\bar{h}^*$ per unit area acting across the boundary $\partial\mathcal{R}^*$ of a region of space $\mathcal{R}^*$ occupied by the body in the present configuration at time t. Also, we assume the existence of an internal surface flux of heat $-h^* = -h^*(\theta^i,t\,;\underset{\sim}{n})$ per unit area across each surface $\partial\mathcal{P}^*$ which is the boundary of an arbitrary part $\mathcal{P}^*$ of $\mathcal{R}^*$. We define the ratio of the heat supply r^* to temperature θ^* as $s^* = s^*(\theta^i,t)$ and call this the external rate of supply of entropy per unit mass. Similarly we define the ratios of $\bar{h}^*$ and h^* to temperature, respectively, as the external rate of surface supply of entropy $\bar{k}^*$ per unit area of $\partial\mathcal{R}^*$ and the internal surface flux of entropy $k^* = k^*(\theta^i,t\,;\underset{\sim}{n})$ per unit area of $\partial\mathcal{P}^*$. Thus,

$$r^* = \theta^* s^* \quad , \quad \bar{h}^* = \theta^* \bar{k}^* \quad , \quad h^* = \theta^* k^* \quad . \tag{7.1}$$

In addition, throughout $\mathcal{R}^*$ we assume the existence of a scalar field $\eta^* = \eta^*(\theta^i,t)$ per unit mass, called the specific entropy and an internal rate of production of entropy $\xi^* = \xi^*(\theta^i,t)$ per unit mass. The contribution of the latter to the internal rate of production of heat is simply $\theta^*\xi^*$ per unit mass.

We recall the balance of entropy in the form

$$\frac{d}{dt}\int_{\mathcal{P}^*} \rho^*\eta^* dv = \int_{\mathcal{P}^*} \rho^*(s^* + \xi^*)dv - \int_{\partial\mathcal{P}^*} k^* da \tag{7.2}$$

for every material volume occupying a part $\mathcal{P}^*$ in the present configuration. It follows that k^* is linear in $\underset{\sim}{n}$, i.e.,

$$k^* = \underset{\sim}{p}^* \cdot \underset{\sim}{n} \quad , \quad \underset{\sim}{p}^* = p^{*i}\underset{\sim}{g}_i \quad , \tag{7.3}$$

where $\underset{\sim}{p}^*$ is the entropy flux vector. Then, from (7.1) and (7.3), $h^* = \theta^*\underset{\sim}{p}^* \cdot \underset{\sim}{n}$

‡No confusion should arise from the notation $\theta^i = (\theta^1,\theta^2,\theta^3)$ for the convected coordinates and the use of the symbol θ in designation of the (three-dimensional) temperature field θ^* and the surface temperatures $\theta,\theta_1,\theta_2,\ldots,\theta_K$ in (7.17).

and we may define the heat flux vector $\underset{\sim}{q}^*$ by

$$\underset{\sim}{q}^* = \theta^* \underset{\sim}{p}^* \ . \tag{7.4}$$

With the help of (7.3), the field equation corresponding to (7.2) is

$$\rho^* \dot{\eta}^* = \rho^*(s^* + \xi^*) - \operatorname{div} \underset{\sim}{p}^* \ , \tag{7.5}$$

where

$$\operatorname{div} \underset{\sim}{p}^* = g^{-\frac{1}{2}} \partial(p^{*k} g^{\frac{1}{2}})/\partial\theta^k \ . \tag{7.6}$$

Now multiply (7.5) by $(\theta^3)^N$ and integrate over an arbitrary part $\mathcal{P}^*$ in the present configuration. After using (7.3) and some straightforward manipulation, we obtain

$$\frac{d}{dt} \int_{\mathcal{P}^*} \rho^* \eta^* (\theta^3)^N dv = \int_{\mathcal{P}^*} \rho^*(s^* + \xi^*)(\theta^3)^N dv + \int_{\mathcal{P}^*} N(\theta^3)^{N-1} p^{*3} dv$$
$$- \int_{\partial\mathcal{P}^*} (\theta^3)^N k^* da \quad (N = 1, 2, \ldots) \ . \tag{7.7}$$

Let an arbitrary material surface $\theta^3 = 0$ occupy a region $\mathcal{P}$ at time t and let $\partial\mathcal{P}$ denote the closed boundary of $\mathcal{P}$. Further, let $\partial\mathcal{P}_n^*$ refer to a part of $\partial\mathcal{P}^*$ specified by the surface (6.5) so that $\partial\mathcal{P}_n^* = \partial\mathcal{P}^* = \partial\mathcal{P}$ on $\theta^3 = 0$, and let $\partial\mathcal{P}_n^{*c} = \partial\mathcal{P}^* - \partial\mathcal{P}_n^*$ stand for the complement of $\partial\mathcal{P}_n^*$ in $\partial\mathcal{P}^*$. Then, for a shell-like region bounded by the surfaces $(6.4)_{1,2}$ and (6.5), from (7.2) and (7.7) with $N = 1, 2, \ldots, K$, we can derive the balance equations (3.5) and (3.6) without introducing any approximations† provided we make the following identifications:

$$\lambda\eta = \int_\alpha^\beta \rho^* \eta^* g^{\frac{1}{2}} d\theta^3 \ , \quad \lambda\eta_N = \int_\alpha^\beta \rho^* \eta^* g^{\frac{1}{2}} (\theta^3)^N \, d\theta^3 \ , \tag{7.8}$$

$$\lambda s = \int_\alpha^\beta \rho^* s^* g^{\frac{1}{2}} d\theta^3 - [k^* g^{\frac{1}{2}} f_\alpha(\theta^3)]_{\theta^3=\alpha} - [k^* g^{\frac{1}{2}} f_\beta(\theta^3)]_{\theta^3=\beta} \ , \tag{7.9}$$

†The details parallel similar developments in Naghdi (1972, Sec. 11).

$$\lambda s_N = \int_\alpha^\beta \rho^* s^* g^{\frac{1}{2}}(\theta^3)^N d\theta^3 - [k^* g^{\frac{1}{2}}(\theta^3)^N \mathfrak{f}_\alpha(\theta^3)]_{\theta^3=\alpha} - [k^* g^{\frac{1}{2}}(\theta^3)^N \mathfrak{f}_\beta(\theta^3)]_{\theta^3=\beta} \quad , \tag{7.10}$$

$$\lambda \xi = \int_\alpha^\beta \rho^* \xi^* g^{\frac{1}{2}} d\theta^3 \quad , \tag{7.11}$$

$$\lambda \xi_N = \lambda \bar{\xi}_N + \int_\alpha^\beta \rho^* \xi^* g^{\frac{1}{2}}(\theta^3)^N d\theta^3 \quad , \quad \lambda \bar{\xi}_N = N \int_\alpha^\beta p^{*3} g^{\frac{1}{2}}(\theta^3)^{N-1} d\theta^3 \quad , \tag{7.12}$$

$$p^\alpha a^{\frac{1}{2}} = \int_\alpha^\beta p^{*\alpha} g^{\frac{1}{2}} d\theta^3 \quad , \quad p_N^\alpha a^{\frac{1}{2}} = \int_\alpha^\beta p^{*\alpha} g^{\frac{1}{2}}(\theta^3)^N d\theta^3 \quad , \tag{7.13}$$

$$k = p^\alpha \nu_\alpha \quad , \quad k_N = p_N^\alpha \nu_\alpha \quad , \tag{7.14}$$

$$h = \theta k \quad , \quad h_N = \theta_N k_N \quad . \tag{7.15}$$

We also recall the three-dimensional equation for the conservation of energy, namely

$$\frac{d}{dt}\int_{\mathcal{P}^*}(\tfrac{1}{2}\,\underset{\sim}{v}^* \cdot \underset{\sim}{v}^* + \epsilon^*)\rho^* dv = \int_{\mathcal{P}^*}(s^*\theta^* + \underset{\sim}{f}^* \cdot \underset{\sim}{v}^*)\rho^* dv + \int_{\partial\mathcal{P}^*}(\underset{\sim}{t} \cdot \underset{\sim}{v}^* - k^*\theta^*) da \quad , \tag{7.16}$$

where ϵ^* is the internal energy density. Suppose in addition to the approximation (6.6) and (6.7) for the displacement and velocity vectors, we adopt the approximation

$$\theta^* = \theta(\theta^\alpha, t) + \sum_{N=1}^{K} (\theta^3)^N \, \theta_N(\theta^\alpha, t) \quad , \quad \theta > 0 \tag{7.17}$$

for the temperature field. Then, for a shell-like region bounded by the surfaces $(6.4)_{1,2}$ and (6.5), from the energy equation (7.16) we can derive the equation of balance of energy (3.12) for a Cosserat surface provided we make the identification

$$\lambda \epsilon = \int_\alpha^\beta \rho^* \epsilon^* g^{\frac{1}{2}} d\theta^3 \quad . \tag{7.18}$$

8. Heat flux vectors and internal energy.

Suppose the three-dimensional shell-like body is in equilibrium with $\underset{\sim}{v}^* = \underset{\sim}{0}$ and all functions are independent of the time. As a part of their thermodynamic restrictions on constitutive equations, Green and Naghdi (1977) have adopted the classical inequality†

$$-\underset{\sim}{q}^* \cdot \underset{\sim}{g}^* \geqq 0 \quad \text{or} \quad -\underset{\sim}{p}^* \cdot \underset{\sim}{g}^* \geqq 0 \ , \tag{8.1}$$

$$\underset{\sim}{g}^* = \operatorname{grad} \theta^*$$

for all time-independent temperature fields. It follows from $(8.1)_2$ that

$$-\int_{\alpha}^{\beta} g^{\frac{1}{2}} \underset{\sim}{p}^* \cdot \underset{\sim}{g}^* d\theta^3 \geqq 0 \ . \tag{8.2}$$

With the approximation (7.17) for θ^* and the help of $(7.12)_2$ and (7.13), it is seen from (8.2) that

$$\sum_{N=1}^{K} \rho\bar{\xi}_N \theta_N + \underset{\sim}{p} \cdot \underset{\sim}{g} + \sum_{N=1}^{K} \underset{\sim}{p}_N \cdot \underset{\sim}{g}_N \leqq 0 \tag{8.3}$$

for all equilibrium displacement and temperature fields. With the above motivation, we add the inequality (8.3) for all equilibrium states to the thermodynamic inequality (5.4) which was derived directly from two-dimensional postulates.

Now suppose that the Cosserat surface $\mathcal{C}$ is at rest with

$$\underset{\sim}{v} = \underset{\sim}{0} \ , \quad \underset{\sim}{w} = \underset{\sim}{0} \tag{8.4}$$

for all time and with the deformation gradient, director and director gradient each constant for all time. Then, by (2.15), ρ is independent of t. In addition we restrict the temperature fields to be spatially homogeneous so that $\theta = \theta(t)$,

† Previously (Green and Naghdi 1970) in the case of an elastic shell, an inequality was derived using $(8.1)_1$. Because of the different thermodynamic restrictions employed here, it is more convenient to start with $(8.1)_2$.

$\theta_N = 0$ $(N = 1,\ldots,K)$. Keeping these conditions in mind, from a combination of (3.10) and (3.13) we have

$$\rho(r + \sum_{N=1}^{K} r_N) - \operatorname{div}(\mathbf{q} + \sum_{N=1}^{K} \mathbf{q}_n) = \rho\dot{\epsilon} \quad . \tag{8.5}$$

In view of (8.4), no mechanical work is supplied to the Cosserat surface $\mathcal{C}$. Hence, using (8.5), the heat supplied to a part $\mathcal{P}$ of the surface during the time interval $t_1 \leqq t \leqq t_2$ is

$$\mathcal{H} = \int_{t_1}^{t_2} [\int_{\mathcal{P}} (r + \sum_{N=1}^{K} r_N)\rho \, d\sigma - \int_{\partial\mathcal{P}} (\mathbf{q} + \sum_{N=1}^{K} \mathbf{q}_N) \cdot \boldsymbol{\nu} \, ds] dt = \int_{\mathcal{P}} \rho\epsilon \Big|_{t_1}^{t_2} \quad . \tag{8.6}$$

Suppose the shell is in thermal equilibrium during some period up to time t_1 with constant internal energy ϵ_1 and constant temperature $\bar{\theta}$. We assume that whenever heat is supplied to the part $\mathcal{P}$ of the shell under the above conditions, the temperature $\theta(t)$ throughout the part will be increased, i.e.,

$$[\theta]_{t_1}^{t_2} > 0 \quad \text{whenever} \quad \mathcal{H} > 0 \quad . \tag{8.7}$$

Provided that $\rho\epsilon$ is continuous and remembering that ρ, which is independent of t, is positive and $\mathcal{P}$ is arbitrary, it follows from (8.6) and (8.7) that

$$\theta(t) - \bar{\theta} > 0 \quad \text{whenever} \quad \epsilon(t) - \epsilon_1 > 0 \tag{8.8}$$

for all $t > t_1$.

9. Symmetries

In the rest of this paper we restrict attention to the situation where $K = 1$ in the earlier part of the paper, and for convenience we set

$$\theta_1 = \phi \quad . \tag{9.1}$$

The inclusion of a second temperature field ϕ allows us to take some account of temperature variations across the shell thickness but the more general situation in which $K > 1$ can be dealt with in a similar way. We also use relative kinematic measures $e_{\alpha\beta}, \gamma_i, \kappa_{i\alpha}$, instead of $a_{\alpha\beta}, d_i, \lambda_{i\alpha}$, where

$$e_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) \quad , \quad \gamma_i = d_i - D_i \quad , \quad \kappa_{i\alpha} = \lambda_{i\alpha} - \Lambda_{i\alpha} \quad . \tag{9.2}$$

We consider the form of the Helmholtz free energy function in (4.17) which is such that the Cosserat surface models the main features of a three-dimensional thin elastic shell which has the following properties in its reference configuration: (i) it is of uniform thickness and normals to the middle surface meet the major surfaces of the shell at points which are equidistant from the middle surface, (ii) it possesses isotropy with a center of symmetry and (iii) it is homogeneous and of constant temperature.

We choose the initial director $\underset{\sim}{D}$ to be specified by

$$\underset{\sim}{D} = D\underset{\sim}{A}_3 \quad , \tag{9.3}$$

where D is a nonzero constant in line with (i) above. Then, from (4.11),

$$\Lambda_{\alpha\beta} = - DB_{\alpha\beta} \quad , \quad \Lambda_{3\alpha} = 0 \quad . \tag{9.4}$$

The Helmholtz function (4.17) may now be replaced by the different function

$$\psi = \tilde{\psi}(e_{\alpha\beta}, \gamma_i, \kappa_{i\alpha}, \theta, \phi \,;\, D, DB_{\alpha\beta}, \underset{\sim}{A}_\gamma) \quad , \tag{9.5}$$

which by virtue of (iii) does not depend explicitly on θ^α. Moreover, recalling

(6.6), our Cosserat surface $\mathcal{C}$ models a three-dimensional body which in its reference state has a position vector $\underset{\sim}{R}^*$ specified by

$$\underset{\sim}{R}^* = \underset{\sim}{R} + \theta^3 D \underset{\sim}{A}_3 \quad . \tag{9.6}$$

At this stage we relax the condition (ii) slightly and assume that the three-dimensional body is transversely isotropic at each point with respect to the normals $\underset{\sim}{A}_3$ to the surface† $\theta^3 = 0$ $(\underset{\sim}{R}^* = \underset{\sim}{R})$. Then, the three-dimensional energy function is form-invariant under the coordinate transformation

$$\bar{\theta}^3 = -\theta^3 \quad , \quad \bar{\theta}^\alpha = \bar{\theta}^\alpha(\theta^1, \theta^2) \quad , \tag{9.7}$$

where $\bar{\theta}^i$ any other set of three-dimensional curvilinear coordinates with $\bar{\theta}^\alpha$ on the middle surface. Corresponding to the first of the transformations in (9.7), we assume that the response function $\tilde{\psi}$ in (9.5) is unaltered in form when

$$D \to -D \quad , \quad \underset{\sim}{d} \to -\underset{\sim}{d} \quad , \quad \phi \to -\phi \quad , \tag{9.8}$$

the other variables remaining unchanged. Thus

$$\tilde{\psi}(e_{\alpha\beta}, -\gamma_i, -\kappa_{i\alpha}, \theta, -\phi \,;\, -D, -DB_{\alpha\beta}, \underset{\sim}{A}_\gamma)$$

$$= \tilde{\psi}(e_{\alpha\beta}, \gamma_i, \kappa_{i\alpha}, \theta, \phi \,;\, D, DB_{\alpha\beta}, \underset{\sim}{A}_\gamma) \quad . \tag{9.9}$$

Since $D \neq 0$, $\tilde{\psi}$ may be reduced to a different functional form and we have

$$\psi = \tilde{\psi}(e_{\alpha\beta}, D\gamma_i, D\kappa_{i\alpha}, \theta, D\phi \,;\, D^2, B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \tag{9.10}$$

if redundant elements are omitted. To complete the specification that the three-dimensional shell is transversely isotropic, we suppose that $\tilde{\psi}$ is an isotropic function with a center of symmetry. Moreover, in view of the second

---

†Complete isotropy is considered in §10, where constitutive coefficients in the linear theory are identified.

part of the condition (i), we assume that the major surfaces of the shell in its reference configuration are specified by $\theta^3 = \pm\frac{1}{2}h$ in (9.6), where h is a constant and $\theta^3 = 0$ is its middle surface. We identify the surface of $\mathcal{C}$ in the reference configuration by $\underset{\sim}{R}^* = \underset{\sim}{R}$ and we then require that ψ is unaltered in form under the transformation

$$\underset{\sim}{A}_3 \to -\underset{\sim}{A}_3 \quad , \quad \underset{\sim}{a}_3 \to -\underset{\sim}{a}_3 \quad . \tag{9.11}$$

The transformation (9.11) implies that

$$\begin{gathered} D \to -D \quad , \quad \gamma_3 \to -\gamma_3 \quad , \quad \gamma_\alpha \to \gamma_\alpha \quad , \quad \kappa_{3\alpha} \to -\kappa_{3\alpha} \quad , \\ \kappa_{\alpha\beta} \to \kappa_{\alpha\beta} \quad , \quad e_{\alpha\beta} \to e_{\alpha\beta} \quad , \quad B_{\alpha\beta} \to -B_{\alpha\beta} \quad , \\ \theta \to \theta \quad , \quad \phi \to \phi \quad . \end{gathered} \tag{9.12}$$

Hence, $\tilde{\psi}$ in (9.10) is subject to the further condition

$$\begin{aligned} &\tilde{\psi}(e_{\alpha\beta}, D\gamma_3, -D\gamma_\alpha, D\kappa_{3\alpha}, -D\kappa_{\alpha\beta}, \theta, -D\phi \,;\, D^2, -B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \\ &= \tilde{\psi}(e_{\alpha\beta}, D\gamma_3, D\gamma_\alpha, D\kappa_{3\alpha}, D\kappa_{\alpha\beta}, \theta, D\phi \,;\, D^2, B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \quad . \end{aligned} \tag{9.13}$$

After imposing the restriction (9.13) it is convenient to make the special choice $D = 1$ for D which implies that h is the thickness of the shell. Thus, (9.10) reduces to the different form

$$\psi = \tilde{\psi}(e_{\alpha\beta}, \gamma_3, \gamma_\alpha, \kappa_{3\alpha}, \kappa_{\alpha\beta}, \theta, \phi \,;\, B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \quad , \tag{9.14}$$

which is isotropic with a center of symmetry and which is also subject to the restriction

$$\begin{aligned} &\tilde{\psi}(e_{\alpha\beta}, \gamma_3, -\gamma_\alpha, \kappa_{3\alpha}, -\kappa_{\alpha\beta}, \theta, -\phi \,;\, -B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \\ &= \tilde{\psi}(e_{\alpha\beta}, \gamma_3, \gamma_\alpha, \kappa_{3\alpha}, \kappa_{\alpha\beta}, \theta, \phi \,;\, B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \quad . \end{aligned} \tag{9.15}$$

A similar discussion may be carried out for the entropy flux vectors $\underset{\sim}{p}_o, \underset{\sim}{p}_1$ and the internal production of entropies ξ, ξ_1, except that all of these functions also depend on $\partial\theta/\partial\theta^\mu$, $\partial\phi/\partial\theta^\mu$. In view of (9.3) and (9.4), the entropy flux vectors may be reduced to functional forms of the type exhibited by $\tilde{\Psi}$ in (9.5), including dependence also on $\partial\theta/\partial\theta^\mu$ and $\partial\phi/\partial\theta^\mu$. Thus

$$\begin{aligned} p_o^\alpha &= \hat{p}_o^\alpha(e_{\alpha\beta}, \gamma_i, \kappa_{i\alpha}, \theta, \phi \; ; \; \frac{\partial\theta}{\partial\theta^\mu}, \frac{\partial\phi}{\partial\theta^\mu} \; ; \; D, DB_{\alpha\beta}, \underset{\sim}{A}_\gamma) \; , \\ p_1^\alpha &= \hat{p}_1^\alpha(e_{\alpha\beta}, \gamma_i, \kappa_{i\alpha}, \theta, \phi \; ; \; \frac{\partial\theta}{\partial\theta^\mu}, \frac{\partial\phi}{\partial\theta^\mu} \; ; \; D, DB_{\alpha\beta}, \underset{\sim}{A}_\gamma) \; . \end{aligned} \tag{9.16}$$

Under the transformations (9.8) and (9.12), we require that

$$\underset{\sim}{p} \cdot \underset{\sim}{g} = p_o^\alpha \partial\theta/\partial\theta^\alpha \;\; , \;\; \underset{\sim}{p}_1 \cdot \underset{\sim}{g}_1 = p_1^\alpha \partial\phi/\partial\theta^\alpha \;\; , \;\; \xi\theta \;\; , \;\; \xi_1\phi \tag{9.17}$$

be unaltered in form. After setting $D = 1$, it follows that p_o^α, p_1^α reduce to the different forms

$$\begin{aligned} p_o^\alpha &= \tilde{p}_o^\alpha(e_{\alpha\beta}, \gamma_3, \gamma_\alpha, \kappa_{3\alpha}, \kappa_{\alpha\beta}, \theta, \phi, \frac{\partial\theta}{\partial\theta^\mu}, \frac{\partial\phi}{\partial\theta^\mu} \; ; \; B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \; , \\ p_1^\alpha &= \tilde{p}_1^\alpha(e_{\alpha\beta}, \gamma_3, \gamma_\alpha, \kappa_{3\alpha}, \kappa_{\alpha\beta}, \theta, \phi, \frac{\partial\theta}{\partial\theta^\mu}, \frac{\partial\phi}{\partial\theta^\mu} \; ; \; B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \; , \end{aligned} \tag{9.18}$$

which are isotropic with a center of symmetry and which are also subject to the restrictions

$$\begin{aligned} &\tilde{p}_o^\alpha(e_{\alpha\beta}, \gamma_3, -\gamma_\alpha, \kappa_{3\alpha}, -\kappa_{\alpha\beta}, \theta, -\phi, \partial\theta/\partial\theta^\mu, -\partial\phi/\partial\theta^\mu \; ; \; -B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \\ &\quad = \tilde{p}_o^\alpha(e_{\alpha\beta}, \gamma_3, \gamma_\alpha, \kappa_{3\alpha}, \kappa_{\alpha\beta}, \theta, \phi, \partial\theta/\partial\theta^\mu, \partial\phi/\partial\theta^\mu \; ; \; B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \; , \end{aligned} \tag{9.19}$$

$$\begin{aligned} &\tilde{p}_1^\alpha(e_{\alpha\beta}, \gamma_3, -\gamma_\alpha, \kappa_{3\alpha}, -\kappa_{\alpha\beta}, \theta, -\phi, \partial\theta/\partial\theta^\mu, -\partial\phi/\partial\theta^\mu \; ; \; -B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \\ &\quad = -\tilde{p}_1^\alpha(e_{\alpha\beta}, \gamma_3, \gamma_\alpha, \kappa_{3\alpha}, \kappa_{\alpha\beta}, \theta, \phi, \partial\theta/\partial\theta^\mu, \partial\phi/\partial\theta^\mu \; ; \; B_{\alpha\beta}, \underset{\sim}{A}_\gamma) \; , \end{aligned} \tag{9.20}$$

with similar forms for ξ, ξ_1, respectively. Finally, each function is also a hemihedral isotropic function of its arguments.

In order to make the above conditions explicit, we limit our attention to the linear theory of a thermoelastic Cosserat surface which is unstressed and at uniform temperature in its reference configuration.† Then, for this theory, the position vector $\underset{\sim}{r}$ and the director $\underset{\sim}{d}$ assume the forms

$$\begin{gathered}\underset{\sim}{r} = \underset{\sim}{R} + \underset{\sim}{u} \quad , \quad \underset{\sim}{d} = D\underset{\sim}{A}_3 + \underset{\sim}{\delta} \quad , \\ \underset{\sim}{u} = u^i \underset{\sim}{A}_i = u_i \underset{\sim}{A}^i \quad , \quad \underset{\sim}{\delta} = \delta_i \underset{\sim}{A}^i = \delta^i \underset{\sim}{A}_i \quad ,\end{gathered} \tag{9.21}$$

where $\underset{\sim}{u}, \underset{\sim}{\delta}$ and their space and time derivatives are small. We also limit our attention to a Cosserat surface which models the main features of a three-dimensional thin elastic shell as described in (i)-(iii) above; and, hence, we set $D = 1$ throughout the rest of the discussion. The linearized relative kinematic measures are

$$e_{\alpha\beta}, \gamma_i, \rho_{i\alpha} \quad , \tag{9.22}$$

where

$$\begin{gathered}e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta}u_3 = \tfrac{1}{2}(\underset{\sim}{u}_{,\alpha} \cdot \underset{\sim}{A}_\beta + \underset{\sim}{u}_{,\beta} \cdot \underset{\sim}{A}_\alpha) \quad , \\ \rho_{\alpha\beta} = \kappa_{\alpha\beta} + B_{\alpha\beta}\gamma_3 \quad , \quad \rho_{3\alpha} = \kappa_{3\alpha} - B^\lambda_\alpha \gamma_\lambda \quad , \\ \gamma_3 = \delta_3 \quad , \quad \gamma_\alpha = \delta_\alpha + u_{3,\alpha} + B^\lambda_\alpha u_\lambda \quad , \quad \rho_{3\alpha} = \gamma_{3,\alpha} \quad , \\ -\rho_{\alpha\beta} = u_{3|\alpha\beta} + B^\lambda_{\alpha|\beta} u_\lambda + B^\lambda_\alpha u_{\lambda|\beta} + B^\lambda_\beta u_{\lambda|\alpha} - B_{\alpha\lambda} B^\lambda_\beta u_3 - \gamma_{\alpha|\beta} \quad ,\end{gathered} \tag{9.23}$$

and where a vertical line now denotes covariant differentiation with respect to the reference surface using Christoffel symbols calculated from $A_{\alpha\beta}$.

We assume henceforth that the reference configuration of the Cosserat surface $\mathcal{C}$ coincides with its initial undeformed configuration and thus the vector fields representing the force, the director force and the internal director force are all zero in the reference configuration. To avoid the introduction of additional notations, we now regard the force vector $\underset{\sim}{N}$ and

†At this point some differences will occur in the developments if we wish to include the effects of surface tension or initial stress in a membrane theory of a surface, but this is straightforward.

the director force vector $\underset{\sim}{M}$ as infinitesimal quantities measured per unit length of curves in the surface $\mathcal{S}$ in the reference configuration of $\mathcal{C}$, so that $\underset{\sim}{\nu}$ is now the outward unit normal to curves in $\mathcal{S}$ and

$$\underset{\sim}{\nu} = \nu_\alpha \underset{\sim}{A}^\alpha \quad , \quad \underset{\sim}{N} = \underset{\sim}{N}^\alpha \nu_\alpha \quad , \quad \underset{\sim}{M} = \underset{\sim}{M}^\alpha \nu_\alpha \ . \tag{9.24}$$

Moreover‡

$$\underset{\sim}{N}^\alpha = N^{\alpha i} \underset{\sim}{A}_i \quad , \quad \underset{\sim}{M}^\alpha = M^{\alpha i} \underset{\sim}{A}_i \quad , \quad \underset{\sim}{m} = m^i \underset{\sim}{A}_i \tag{9.25}$$

and the equations of motion in tensor components become

$$\begin{aligned} N^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha V^\alpha + \rho f^\beta &= \rho(\dot{\underset{\sim}{v}} + k^1 \dot{\underset{\sim}{w}}) \cdot \underset{\sim}{A}^\beta \ , \\ V^\alpha{}_{|\alpha} + B_{\alpha\beta} N^{\alpha\beta} + \rho f^3 &= \rho(\dot{\underset{\sim}{v}} + k^1 \dot{\underset{\sim}{w}}) \cdot \underset{\sim}{A}_3 \ , \end{aligned} \tag{9.26}$$

and

$$\begin{aligned} M^{\alpha\beta}{}_{|\alpha} - V^\beta + \rho \ell^\beta &= \rho(k^1 \dot{\underset{\sim}{v}} + k^2 \dot{\underset{\sim}{w}}) \cdot \underset{\sim}{A}^\beta \ , \\ M^{\alpha 3}{}_{|\alpha} - V^3 + \rho \ell^3 &= \rho(k^1 \dot{\underset{\sim}{v}} + k^2 \dot{\underset{\sim}{w}}) \cdot \underset{\sim}{A}_3 \ , \end{aligned} \tag{9.27}$$

where

$$\begin{aligned} N^{\beta 3} = V^\beta = m^\beta + B^\beta_\alpha M^{\alpha 3} \quad &, \quad V^3 = m^3 - B_{\alpha\beta} M^{\alpha\beta} \ , \\ \underset{\sim}{f} = f^i \underset{\sim}{A}_i \quad &, \quad \underset{\sim}{\ell} = \ell^i \underset{\sim}{A}_i \end{aligned} \tag{9.28}$$

and ρ denotes the reference density.

Turning to the temperature variables we replace θ by $\bar{\theta} + \theta$, where $\Theta = \bar{\theta}$ is

‡The order of indices $N^{\alpha i}, M^{\alpha i}$ has been used by Naghdi (1972) and in many earlier papers in order to correspond to a usual notation in shell theory. In the first paper by Green, Naghdi and Wainwright (1965) the opposite order $N^{i\alpha}, M^{i\alpha}$ was used and this would be more in line with the coordinate free notations of the form $\underset{\sim}{N} = \underset{\sim}{\bar{N}}\,\underset{\sim}{\nu} = \bar{N}^{i\alpha} \nu_\alpha \underset{\sim}{A}_i$ sometimes employed in the current literature.

constant, and assume that θ and ϕ and their derivatives are small. Then, with

$$\underset{\sim}{p} = p^{\alpha} \underset{\sim}{A}_{\alpha} \quad , \quad \underset{\sim}{p}_1 = p_1^{\alpha} \underset{\sim}{A}_{\alpha} \quad , \tag{9.29}$$

the equations of balance of entropy can be expressed as

$$\begin{aligned} \rho\dot{\eta} &= \rho(s + \xi) - p^{\alpha}{}_{|\alpha} \quad , \\ \rho\dot{\eta}_1 &= \rho(s_1 + \xi_1) - p_1^{\alpha}{}_{|\alpha} \quad . \end{aligned} \tag{9.30}$$

Also, from (3.3) and (3.9), retaining only first order terms we have

$$\underset{\sim}{q} = \overline{\theta}\underset{\sim}{p} \quad , \quad \underset{\sim}{q}_1 = 0 \quad , \quad r = \overline{\theta}s \quad , \quad r_1 = 0 \quad . \tag{9.31}$$

We recall from the development of §4 that most of the constitutive response functions in the (nonlinear) thermoelastic theory of a Cosserat surface can be expressed in terms of the Helmholtz free energy response function and that this is mainly due to the use of the energy equation as an identity for all thermomechanical processes. A similar situation exists, of course, in the linearized theory and leads to

$$\psi = \tilde{\psi}(e_{\alpha\beta}, \gamma_i, \rho_{i\alpha}, \theta, \phi \,;\, B_{\alpha\beta}, \underset{\sim}{A}_{\alpha}) \tag{9.32}$$

and

$$\begin{aligned} N'^{\alpha\beta} &= \tfrac{1}{2}\rho\left(\frac{\partial\tilde{\psi}}{\partial e_{\alpha\beta}} + \frac{\partial\tilde{\psi}}{\partial e_{\beta\alpha}}\right) \quad , \quad M^{\alpha i} = \rho\,\frac{\partial\tilde{\psi}}{\partial\rho_{i\alpha}} \quad , \quad V^i = \rho\,\frac{\partial\tilde{\psi}}{\partial\gamma_i} \quad , \\ \eta &= -\frac{\partial\tilde{\psi}}{\partial\theta} \quad , \quad \eta_1 = -\frac{\partial\tilde{\psi}}{\partial\phi} \quad , \end{aligned} \tag{9.33}$$

where

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + B^{\beta}_{\lambda}M^{\lambda\alpha} \quad . \tag{9.34}$$

Further, if the various forces, director forces and the entropies all vanish in the reference configuration, the response function $\tilde{\psi}$ in (9.26) is a quadratic function of $e_{\alpha\beta}, \gamma_i, \rho_{i\alpha}, \theta, \phi$. This function is to be a hemihedal isotropic function which satisfies the condition (9.15).

In order to obtain an explicit form for $\tilde{\psi}$, it is convenient to introduce the invariant surface tensors

$$\underset{\sim}{E} = e_{\alpha\beta}\underset{\sim}{A}^{\alpha}\otimes\underset{\sim}{A}^{\beta} \quad , \quad \underset{\sim}{B} = B_{\alpha\beta}\underset{\sim}{A}^{\alpha}\otimes\underset{\sim}{A}^{\beta} \quad ,$$

$$\underset{\sim}{J} = \rho_{\alpha\beta}\underset{\sim}{A}^{\alpha}\otimes\underset{\sim}{A}^{\beta} \quad , \quad \underset{\sim}{K} = \rho_{3\alpha}\rho_{3\beta}\underset{\sim}{A}^{\alpha}\otimes\underset{\sim}{A}^{\beta} \quad , \tag{9.35}$$

$$\underset{\sim}{P} = \rho_{3\alpha}\gamma_{\beta}\underset{\sim}{A}^{\alpha}\otimes\underset{\sim}{A}^{\beta} \quad , \quad \underset{\sim}{R} = \gamma_{\alpha}\gamma_{\beta}\underset{\sim}{A}^{\alpha}\otimes\underset{\sim}{A}^{\beta} \quad ,$$

where $\underset{\sim}{E}, \underset{\sim}{B}, \underset{\sim}{K}$ and $\underset{\sim}{R}$ are symmetric. Then,

$$\psi = \tilde{\psi}(\underset{\sim}{E},\underset{\sim}{J},\underset{\sim}{K},\underset{\sim}{P},\underset{\sim}{R},\gamma_3,\theta,\phi\,;\,\underset{\sim}{B}) \quad , \tag{9.36}$$

where $\tilde{\psi}$ is an isotropic scalar invariant function of its arguments, such that it is a quadratic form in the variables $e_{\alpha\beta}, \gamma_i, \rho_{i\alpha}, \theta, \phi$ and

$$\tilde{\psi}(\underset{\sim}{E},-\underset{\sim}{J},\underset{\sim}{K},-\underset{\sim}{P},\underset{\sim}{R},\gamma_3,\theta,-\phi\,;\,-\underset{\sim}{B}) = \tilde{\psi}(\underset{\sim}{E},\underset{\sim}{J},\underset{\sim}{K},\underset{\sim}{P},\underset{\sim}{R},\gamma_3,\theta,\phi\,;\,\underset{\sim}{B}) \quad . \tag{9.37}$$

It follows (Rivlin 1955) that $\tilde{\psi}$ is a linear function of the following 39 invariants

$$\gamma_3^2 \; , \; \gamma_3 \operatorname{tr} \underset{\sim}{E} \; , \; (\operatorname{tr} \underset{\sim}{E})^2 \; , \; (\operatorname{tr} \underset{\sim}{J})^2 \; , \; \operatorname{tr} \underset{\sim}{E}^2 \; , \; \operatorname{tr} \underset{\sim}{J}^2 \quad ,$$

$$\operatorname{tr} \underset{\sim}{J}\underset{\sim}{J}^T \; , \; \operatorname{tr} \underset{\sim}{K} \; , \; \operatorname{tr} \underset{\sim}{R} \quad , \tag{9.38}$$

$$\theta^2 \ , \ \theta\gamma_3 \ , \ \theta \operatorname{tr} \underset{\sim}{E} \ ,$$

$$\phi^2 \ , \ \phi \operatorname{tr} \underset{\sim}{J} \ , \tag{9.39}$$

$$\gamma_3 \operatorname{tr} \underset{\sim}{J}\underset{\sim}{B} \ , \ \gamma_3 \operatorname{tr} \underset{\sim}{E}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{B} \ , \ \gamma_3 \operatorname{tr} \underset{\sim}{J} \operatorname{tr} \underset{\sim}{B} \ ,$$

$$\operatorname{tr} \underset{\sim}{E} \operatorname{tr} \underset{\sim}{J}\underset{\sim}{B} \ , \ \operatorname{tr} \underset{\sim}{E} \operatorname{tr} \underset{\sim}{J} \operatorname{tr} \underset{\sim}{B} \ , \ \operatorname{tr} \underset{\sim}{E} \operatorname{tr} \underset{\sim}{E}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{B} \ ,$$

$$(\operatorname{tr} \underset{\sim}{J}\underset{\sim}{B})^2 \ , \ \operatorname{tr} \underset{\sim}{J}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{J} \operatorname{tr} \underset{\sim}{B} \ , \ \operatorname{tr} \underset{\sim}{E}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{J}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{B} \ , \tag{9.40}$$

$$\operatorname{tr} \underset{\sim}{E}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{J} \ , \ (\operatorname{tr} \underset{\sim}{E}\underset{\sim}{B})^2 \ , \ \operatorname{tr} \underset{\sim}{E}\underset{\sim}{J} \operatorname{tr} \underset{\sim}{B} \ ,$$

$$\operatorname{tr} \underset{\sim}{K}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{B} \ , \ \operatorname{tr} \underset{\sim}{R}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{B} \ , \ \operatorname{tr} \underset{\sim}{P} \operatorname{tr} \underset{\sim}{B} \ ,$$

$$\operatorname{tr} \underset{\sim}{E}\underset{\sim}{J}\underset{\sim}{B} \ , \ \operatorname{tr} \underset{\sim}{B}\underset{\sim}{J}\underset{\sim}{J}^T \operatorname{tr} \underset{\sim}{B} \ ,$$

$$\theta \operatorname{tr} \underset{\sim}{J}\underset{\sim}{B} \ , \ \theta \operatorname{tr} \underset{\sim}{J} \operatorname{tr} \underset{\sim}{B} \ , \ \theta \operatorname{tr} \underset{\sim}{E}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{B} \ , \ \theta\phi \operatorname{tr} \underset{\sim}{B} \ , \tag{9.41}$$

$$\phi\gamma_3 \operatorname{tr} \underset{\sim}{B} \ , \ \phi \operatorname{tr} \underset{\sim}{E} \operatorname{tr} \underset{\sim}{B} \ , \ \phi \operatorname{tr} \underset{\sim}{J}\underset{\sim}{B} \operatorname{tr} \underset{\sim}{B} \ , \ \phi \operatorname{tr} \underset{\sim}{E}\underset{\sim}{B} \ ,$$

with coefficients which depend on

$$(\operatorname{tr} \underset{\sim}{B})^2 \ , \quad \operatorname{tr} \underset{\sim}{B}^2 \ . \tag{9.42}$$

Constitutive equations for $N'^{\alpha\beta}, M^{\alpha i}, V^i, \eta_o, \eta_1$ (and hence $N^{\alpha\beta}$) may be obtained from (9.33), (9.34) and (9.38) to (9.41).

It remains to consider the constitutive equations for $\underset{\sim}{p}_o, \underset{\sim}{p}_1, \xi, \xi_1, \bar{\xi}_1$. In determining these we must also use the identity (4.8) which here becomes

$$\rho(\bar{\theta}+\theta)\xi + \rho\phi\,\xi_1 + \underset{\sim}{p}\cdot\underset{\sim}{g} + \underset{\sim}{p}_1\cdot\underset{\sim}{g}_1 = 0 \ , \tag{9.43}$$

and the inequality (8.3) which here reduces to

$$\rho\phi\,\bar{\xi}_1 + \underset{\sim}{p}\cdot\underset{\sim}{g} + \underset{\sim}{p}_1\cdot\underset{\sim}{g}_1 \leq 0 \ . \tag{9.44}$$

In the linearized theory, we assume that $\underset{\sim}{p},\underset{\sim}{p}_1$ and ξ,ξ_1 are linear hemihedral isotropic functions of degree one in $e_{\alpha\beta},\gamma_i,\rho_{i\alpha},\theta,\phi,\partial\theta/\partial\theta^\mu,\partial\phi/\partial\theta^\mu$ with coefficients which depend on $A_{\alpha\beta},B_{\alpha\beta}$ such that $\underset{\sim}{p},\underset{\sim}{p}_1$ and ξ,ξ_1 are subject to the conditions (9.19) and (9.20), respectively. If we retain only linear terms in (9.43), we see that[†]

$$\xi = 0 \quad . \tag{9.45}$$

Moreover, with the help of (9.44), it follows that

$$\begin{gathered}\underset{\sim}{p}_o = -a_o\underset{\sim}{g} - b_o\underset{\sim}{B}\underset{\sim}{g}_1 \quad , \quad \underset{\sim}{p}_1 = -a_1\underset{\sim}{B}\underset{\sim}{g} - b_1\underset{\sim}{g}_1 \quad , \\ \rho\bar{\xi}_1 = -b_2\phi \quad , \quad \rho\xi_1 = -a_3\theta\,\mathrm{tr}\,\underset{\sim}{B} - b_3\phi - c_3\mathrm{tr}\,\underset{\sim}{B}\underset{\sim}{E} - d_3\,\mathrm{tr}\,J \quad ,\end{gathered} \tag{9.46}$$

where

$$\underset{\sim}{g} = (\partial\theta/\partial\theta^\alpha)\underset{\sim}{A}^\alpha \quad , \quad \underset{\sim}{g}_1 = (\partial\phi/\partial\theta^\alpha)\underset{\sim}{A}^\alpha \quad , \tag{9.47}$$

$a_o,\ldots,d_3$ are scalar functions of the invariants (9.42) and

$$b_2 \geq 0 \;,\; a_o \geq 0 \;,\; b_1 \geq 0 \;,\; a_o\underset{\sim}{g}\cdot\underset{\sim}{g} + (a_1+b_o)\underset{\sim}{B}\underset{\sim}{g}\cdot\underset{\sim}{g}_1 + b_1\underset{\sim}{g}_1\cdot\underset{\sim}{g}_1 \geq 0 \tag{9.48}$$

for all $\underset{\sim}{g},\underset{\sim}{g}_1$.

[†]Equation (9.43) then determines the second order terms in ξ, but these are not required here.

10. Thermoelastic plate

When the reference surface $\mathcal{S}$ is a plane so that $\underset{\sim}{B} = \underset{\sim}{0}$, the results of the previous section simplify. In particular, we have

$$\begin{aligned}
2\rho\psi = & [\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]e_{\alpha\beta}e_{\gamma\delta} + \alpha_4(\gamma_3)^2 \\
& + [\alpha_5 A^{\alpha\beta}A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma}A^{\beta\delta} + \alpha_7 A^{\alpha\delta}A^{\beta\gamma}]\rho_{\alpha\beta}\rho_{\gamma\delta} \\
& + \alpha_3 A^{\alpha\beta}\gamma_\alpha\gamma_\beta + \alpha_8 A^{\alpha\beta}\rho_{3\alpha}\rho_{3\beta} + 2\alpha_9 A^{\alpha\beta}e_{\alpha\beta}\gamma_3 \\
& + 2\beta_o\gamma_3\theta + 2\beta_1 A^{\alpha\beta}e_{\alpha\beta}\theta + 2\beta_2 A^{\alpha\beta}\rho_{\alpha\beta}\phi \\
& + \beta_3\theta^2 + \beta_4\phi^2 \quad ,
\end{aligned} \tag{10.1}$$

and

$$\begin{gathered}
\rho\eta = -(\beta_o\gamma_3 + \beta_1 A^{\alpha\beta}e_{\alpha\beta} + \beta_3\theta) \quad , \quad \rho\eta_1 = -[\beta_2 A^{\alpha\beta}\rho_{\alpha\beta} + \beta_4\phi] \quad , \\
\underset{\sim}{p} = -a_o\underset{\sim}{g} \quad , \quad \underset{\sim}{p}_1 = -b_1\underset{\sim}{g}_1 \quad , \\
\xi = 0 \quad , \quad \rho\bar{\xi}_1 = -b_2\phi \quad , \quad \rho\xi_1 = -b_3\phi - d_3\,\mathrm{tr}\,J \quad ,
\end{gathered} \tag{10.2}$$

where the coefficients in (10.1)-(10.2) are constants.

We now suppose that the Cosserat plate characterized by (10.1) and (10.2) and other relevant linear constitutive equations <u>models</u> the small deformation of a (three-dimensional) plate of initial constant thickness h and of an elastic material with the following properties: Young's modulus E, Poisson's ratio ν, conductivity κ, specific heat at constant volume c, uniform density ρ^*, coefficient of linear expansion α^* and initial constant temperature $\bar{\theta}$. Let $\underset{\sim}{R}^*$ be the position vector of any point of the (three-dimensional) plate in its reference state given by

$$\underset{\sim}{R}^* = \underset{\sim}{R} + \theta^3\underset{\sim}{A}_3 \quad , \tag{10.3}$$

where $\underset{\sim}{R}$ is the position vector to the middle plane ($\theta^3 = 0$) of the plate, and let the major surfaces of the plate be specified by $\theta^3 = \pm\frac{1}{2}h$. In these circumstances, specific values have been suggested for most of the constitutive coefficients occurring in (10.1) and (10.2) [see, e.g., Naghdi 1972]. These values are

$$\alpha_1 = \alpha_9 = \frac{\nu(1-\nu)}{1-2\nu} C \ , \quad \alpha_2 = \tfrac{1}{2}(1-\nu)C \ , \quad \alpha_4 = \frac{(1-\nu)^2}{1-2\nu} C \ ,$$
$$\alpha_5 = \nu B \ , \quad \alpha_6 = \alpha_7 = \tfrac{1}{2}(1-\nu)B \ , \quad \alpha_3 = \frac{5}{6}\mu h \ , \quad \alpha_8 = \frac{7}{120}\mu h^3 \ , \tag{10.4}$$

where

$$C = \frac{Eh}{1-\nu^2} \ , \quad B = \frac{Eh^3}{12(1-\nu^2)} \ , \quad \mu = \frac{E}{2(1+\nu)} \tag{10.5}$$

and

$$\beta_o = \beta_1 = -\frac{Eh\alpha^*}{1-2\nu} \ , \quad \beta_2 = -\frac{Eh^3\alpha^*}{12(1-2\nu)} \ . \tag{10.6}$$

In order to obtain suitable values for the remaining coefficients we make use of the results in §7 where all quantities now refer to the reference state (10.3) and we choose

$$\alpha = -\tfrac{1}{2}h \ , \quad \beta = \tfrac{1}{2}h \ . \tag{10.7}$$

For a linear elastic solid it follows from previous papers of Green and Naghdi (1970,1977) that if only linear terms are retained, then

$$\xi^* = 0 \ , \quad \underset{\sim}{p}^* = -(\kappa/\bar{\theta})\operatorname{grad}\theta^* \ ,$$
$$\rho^*\eta^* = \frac{E\alpha^*}{1-2\nu}\operatorname{div}\underset{\sim}{u}^* + \frac{\rho^* c\theta^*}{\bar{\theta}} \ , \tag{10.8}$$

where $\underset{\sim}{u}^*$ is the three-dimensional displacement vector. We use the formulae (7.9), (7.12), (7.13), (7.14), (10.8) and the approximation (7.17) for θ^* to obtain the following values for the coefficients $\beta_3, \beta_4, d_3, a_o, b_1, b_2, b_3$:

$$\beta_3 = -\frac{\rho^* ch}{\bar\theta} \ , \quad \beta_4 = -\frac{\rho^* ch^3}{12\bar\theta} \ , \quad d_3 = 0 \ ,$$
$$b_2 = b_3 = \frac{\kappa h}{\bar\theta} \ , \quad a_o = \frac{\kappa h}{\bar\theta} \ , \quad b_1 = \frac{\kappa h^3}{12\bar\theta} \ . \tag{10.9}$$

We adopt these values for the Cosserat surface.

To complete the theory, we need to specify values for the assigned force, the assigned director force and the assigned supplies of entropy, and to this end we consider two problems below. For a discussion of values of the assigned force and assigned director force in the case of either problem, the reader is referred to Naghdi (1972, Ch. E). For the purpose of specifying the assigned entropy supplies, we suppose in the first problem that our direct theory models a (three-dimensional) plate in free space with the ambient temperature of the surroundings at the major surfaces of the plate having the values

$$\theta^* = \theta_+ \ \text{at} \ \theta^3 = \tfrac{1}{2}h + 0 \ , \quad \theta^* = \theta_- \ \text{at} \ \theta^3 = -\tfrac{1}{2}h - 0 \ . \tag{10.10}$$

The temperature across the faces is discontinuous and we assume a radiation type surface condition of the form

$$k^* = K(\theta^* - \theta_+) \ \text{at} \ \theta_3 = \tfrac{1}{2}h \ ,$$
$$k^* = K(\theta^* - \theta_-) \ \text{at} \ \theta_3 = -\tfrac{1}{2}h \ , \tag{10.11}$$

where K is a constant. There are no entropy sources in the plate so that $s^* = 0$. With the help of (6.8), (7.10), (7.11), (10.3), (10.7) and (10.11), we find that

$$\rho s_o = -K(2\theta - \theta_+ - \theta_-) \ , \quad \rho s_1 = -K[\tfrac{1}{2}h^2\phi - \tfrac{1}{2}h(\theta_+ - \theta_-)] \ . \tag{10.12}$$

Collecting together values of $\eta, \eta_1, s_o, s_1, \xi, \xi_1, p^\alpha, p_1^\alpha$, from (10.2), (10.6), (10.9) and (10.12) and substituting these into (9.30) yields the following differential equations for temperatures θ and ϕ:

$$-\bar{\theta}\frac{\partial}{\partial t}\left[\frac{Eh\alpha^*}{1-2\nu}(\gamma_3 + A^{\alpha\beta}e_{\alpha\beta}) + \frac{\rho^* ch\theta}{\bar{\theta}}\right] + \kappa h\nabla^2\theta - H(2\theta - \theta_+ - \theta_-) = 0 \quad , \tag{10.13}$$

$$-\bar{\theta}\frac{\partial}{\partial t}\left[\frac{Eh^3\alpha^*}{12(1-2\nu)}A^{\alpha\beta}\rho_{\alpha\beta} + \frac{\rho^* ch^3\phi}{12\bar{\theta}}\right] + \frac{\kappa h^3}{12}\nabla^2\phi - H[\tfrac{1}{2}h^2\phi - \tfrac{1}{2}h(\theta_+ - \theta_-)] - \kappa h\phi = 0 \ , \tag{10.14}$$

where $H = K\bar{\theta}$ and ∇^2 is the two-dimensional Laplacian. Apart from notation these results agree with those obtained from a three-dimensional approximation by Green and Naghdi (1970,1971).† The inequality (9.48) merely yields the condition $\kappa \geqq 0$ which is already known from three-dimensional theory. If, from the outset, we had restricted the discussion to a theory with one temperature field θ, we would only have the one temperature equation (10.13) and we would obtain no information about the temperature variation across the plate.

In the second problem we assume that the region $-\frac{1}{2}h \leqq \theta^3 \leqq \frac{1}{2}h$ is a material interface between two other media in the regions $\theta^3 \geqq \frac{1}{2}h$ and $\theta^3 \leqq \frac{1}{2}h$. The temperature and heat flux (and hence also the entropy flux) are continuous across the boundaries. We again suppose that the temperatures in the surroundings of the interface at its boundaries have the values (10.10) and we now suppose that the entropy fluxes at these boundaries are

$$k^* = k_+ \ \text{ at } \ \theta^3 = \tfrac{1}{2}h + 0 \quad , \quad k^* = k_- \ \text{ at } \ \theta^3 = -\tfrac{1}{2}h - 0 \quad . \tag{10.15}$$

Because of the continuity of temperature and in view of the representation (7.17) in the interface with $K = 1$, we choose

$$\theta_+ = \theta + \tfrac{1}{2}h\phi \quad , \quad \theta_- = \theta - \tfrac{1}{2}h\phi \ . \tag{10.16}$$

Since k^* is continuous at $\theta^3 = \pm\frac{1}{2}h$, it follows from (6.8), (7.9), (7.10), (10.3) and (10.15) that

---

†Except for a misprint in Eq. (9.27) in 1970 and Eq. (4.5) in 1971. The value of β_1 in these papers should read: $\beta_1 = -Eh_0^3\alpha/12(1-2\eta)$ with η being Poisson's ratio.

$$\rho s = -k_+ - k_- \quad , \quad \rho s_1 = -\tfrac{1}{2}h(k_+ - k_-) \tag{10.17}$$

when $s^* = 0$. The values of k_+ and k_- will depend on the solutions of the field equations for each region surrounding the plate and will be expressible as functions of θ_+ and θ_-, respectively, and can then be written in terms of θ and ϕ by (10.16). If the bounding regions are elastic the heat conduction vectors for each will be subject to the usual restrictions which will affect the values of k_+, k_- but these restrictions do not arise from any inequality associated with the interface. Equations for the temperatures θ, ϕ are then given by (9.30), (10.2), (10.9) and (10.17) and only differ from (10.13) and (10.14) as far as the terms in ρs and ρs_1 are concerned.

It is clear that results for a membrane interface in which the director is omitted from the kinematic variables can be obtained as a special case. If two temperature fields are admitted we still have two equations for the temperatures which for the second problem discussed above reduce to

$$\frac{\partial}{\partial t}\left[\frac{Eh\alpha^*\bar{\theta}}{1-2\nu} A^{\alpha\beta} e_{\alpha\beta} + \rho^* ch\theta\right] - \kappa h \nabla^2 \theta + k_+ + k_- = 0 \quad , \tag{10.18}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho^* ch^2 \phi}{6}\right) - \frac{\kappa h^2}{6} \nabla^2 \phi + k_+ - k_- + 2\kappa\phi = 0 \tag{10.19}$$

for a membrane interface. There is a considerable difference between such equations and those of Murdoch (1976). If ϕ is taken to be zero then, from (10.16) and (10.19), we see that the theory only applies to problems for which $\theta_+ = \theta_- = \theta$, $k_+ = k_-$.

Acknowledgment. The work of one of us (P.M.N.) was supported by the U.S. Office of Naval Research under Contract N00014-75-C-0148, Project NR 064-436, with the University of California, Berkeley.

<u>References</u>

Carroll, M. M. & Naghdi, P. M. 1972 The influence of the reference geometry on the response of elastic shells. Arch. Rational Mech. Anal. <u>48</u>, 302.

Green, A. E. & Naghdi, P. M. 1970 Non-isothermal theory of rods, plates and shells. Int. J. Solids and Structures <u>6</u>, 209.

Green, A. E. & Naghdi, P. M. 1971 On uniqueness in the linear theory of elastic shells and plates. J. Mécanique <u>10</u>, 251.

Green, A. E. & Naghdi, P. M. 1976 Directed fluid sheets. Proc. Roy. Soc. Lond. A <u>347</u>, 447.

Green, A. E. & Naghdi, P. M. 1977 On thermodynamics and the nature of the second law. Proc. Roy. Soc. Lond. A <u>357</u>, 253.

Green, A. E. & Naghdi, P. M. 1978 The second law of thermodynamics and cyclic processes. J. Appl. Mech. To appear.

Green, A. E., Naghdi, P. M. & Wainwright, W. L. 1965 A general theory of a Cosserat surface. Arch. Rational Mech. Anal. <u>20</u>, 287.

Gurtin, M. E. & Murdoch, A. I. 1974 A continuum theory of elastic material surfaces. Arch. Rational Mech. Anal. <u>57</u>, 291.

Gurtin, M. E. & Murdoch, A. I. 1975 Addenda to our paper on a continuum theory of elastic material surfaces. Arch. Rational Mech. Anal. <u>59</u>, 389.

Murdoch, A. I. 1976a A thermodynamical theory of elastic material interfaces. Quart. J. Mech. Appl. Math. <u>29</u>, 245.

Murdoch, A. I. 1976b On the entropy inequality for material surfaces. J. Appl. Math. Phys. (ZAMP) <u>27</u>, 599.

Naghdi, P. M. 1972 The theory of shells and plates. Flügge's Handbuch der Phy. Vol. VIa/2 (edited by C. Truesdell), 425.

Naghdi, P. M. 1964 On the nonlinear thermoelastic theory of shells. Proc. IASS Symp. on non-classical shell problems (Warsaw 1963), p. 5.

Rivlin, R. S. 1955 Further remarks on the stress-deformation relations for isotropic materials. J. Rational Mech. Anal. <u>4</u>, 681.

Zhilin, P. A. 1976 Mechanics of deformable directed surfaces. Int. J. Solids and Structures <u>12</u>, 635.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: UCB/AM-78-4

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle): On Thermal Effects in the Theory of Shells

5. TYPE OF REPORT & PERIOD COVERED: Technical Report

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): A. E. Green and P. M. Naghdi

8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0148

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Department of Mechanical Engineering
University of California
Berkeley, California 94720

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR 064-436

11. CONTROLLING OFFICE NAME AND ADDRESS:
Structural Mechanics Branch
Office of Naval Research
Arlington, Virginia 22217

12. REPORT DATE: September 1978

13. NUMBER OF PAGES: 50

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Thermomechanics of shells, direct approach, theory of a Cosserat surface, temperature changes along the shell thickness, thermodynamical restrictions, nonlinear thermoelastic response, linear theory of thermoelastic plates, determination of constitutive coefficients.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper is concerned with thermomechanics of thin shells by a direct approach based on the theory of a Cosserat surface comprising a two-dimensional surface and a single director attached to every point of the surface. In almost all previous developments of the thermo-mechanical theory of shells by direct approach, only one temperature field has been admitted. This allows for the characterization of temperature changes along some reference surface, such as

(continued)

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20 Abstract (continued)

the middle surface, of the (three-dimensional) shell-like body, but not for temperature changes along the shell thickness. A main purpose of the present study is to incorporate the latter effect into the theory; and, in the context of the theory of a Cosserat surface, this is achieved by a recent approach to thermomechanics (Green and Naghdi 1977) which provides a natural way of introducing two (or more) temperature fields at each material point of the surface. Apart from full discussion of thermomechanics of shells and thermodynamical restrictions arising from the second law of thermodynamics for shells, attention is given to a discussion of symmetries (including material symmetries) and thermal effects in the nonlinear theory of elastic shells with detailed discussion of the linear theory of elastic plates.